Running-Time Lower Bound 101

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(Based on Works that are not totally mine.)

This Talk

- The sole purpose of this talk is to advertise & recruit (prospective) PhD/Master students & Interns.

Rank institutions in the world ▾ by publications from 2008 ▾ to 2018 ▾

All Areas [off | on]

AI [off | on]

- ▶ Artificial intelligence ☐
- ▶ Computer vision ☐
- ▶ Machine learning & data mining ☐
- ▶ Natural language processing ☐
- ▶ The Web & information retrieval ☐

Systems [off | on]

- ▶ Computer architecture ☐
- ▶ Computer networks ☐
- ▶ Computer security ☐
- ▶ Databases ☐
- ▶ Design automation ☐
- ▶ Embedded & real-time systems ☐
- ▶ High-performance computing ☐
- ▶ Mobile computing ☐
- ▶ Measurement & perf. analysis ☐
- ▶ Operating systems ☐
- ▶ Programming languages ☐
- ▶ Software engineering ☐

Theory [off | on]

- ▶ Algorithms & complexity ☑
- ▶ Cryptography ☐
- ▶ Logic & verification ☐

Interdisciplinary Areas [off | on]

- ▶ Comp. bio & bioinformatics ☐
- ▶ Computer graphics ☐
- ▶ Economics & computation ☑

#	Institution	Count	Faculty
1	▶ Carnegie Mellon University	37.4	15
2	▶ Tel Aviv University	35.8	17
3	▶ Stanford University	28.2	18
4	▶ Cornell University	24.4	16
5	▶ Harvard University	21.6	10
6	▶ Technion	21.4	27
7	▶ Massachusetts Institute of Technology	21.2	22
8	▶ University of Pennsylvania	16.6	8
9	▶ Hebrew University of Jerusalem	15.9	17
10	▶ University of Southern California	15.5	8
11	▶ University of Michigan	15.1	11
12	▶ Duke University	14.6	5
13	▼ Shanghai Univ. of Finance and Economics	12.2	5

Faculty	# Pubs	Adj. #
Pinyan Lu THEORY,ECOM	31	11.7
Bundit Laekhanukit THEORY	14	5.1
Nick Gravin THEORY	13	4.1
Tsz Chiu Kwok THEORY	5	1.4
Zihe Wang AI,ECOM	4	1.7

#	Institution	Count	Faculty
14	▶ Columbia University	12.1	9
15	▶ Princeton University	11.6	11
16	▶ Rutgers University	11.4	10
17	▶ University of Washington	10.7	8

The Real Talk

- Address the question of why we cannot break running-time barriers of some classical problems.

Textbook Problems

that we have already hit the barriers

- SAT, k-SUM, k- Clique, Closest-Pair,
- Longest-Common Subsequence,
- All-Pair-Shortest-Paths,
- Maximum-Flow

Classical Techniques

difficult and have a lot of limitations

- Decision Tree Model

Sorting, 2D Closest-Pair, Shortest-Path

limited to comparison based machines

- Circuit Models

k-Cliques, k-Set cover, k-Dominating Set

limited to some class of circuits

- Adversarial Models

Data Structures, Online Algorithms

limited to some classes of algorithms

Modern Techniques

give up improving the known algorithms

- Assume that the known algorithm are the best possible ones!
- Deduce running-time lower-bounds from these given up problems!

Popular Hypotheses

sources of hardness for others problems

- **ETH:** SAT has no $2^{o(n)}$-time algorithm.
- **SETH:** SAT has no $2^{(1-\varepsilon)n}$-time algorithm.
- **k-SUM:** Find k integers from $S \subseteq [-n^2, n^2]$ that sum to zero requires $n^{k/2-\varepsilon}$-time.
- **FPT ≠ W[1]:** k-Clique has no T(k) poly(n)-time algorithm.
- **ATSP:** All-Pair Shortest-Path has no $n^{(3-\varepsilon)}$-time algorithm.

Warming-Ups

lower-bounds based on SAT (SETH)

- Reducing NP-Complete problems to P!

SAT → k-Set-Cover & (Bichromatic) Closest Pairs.

Warming-Ups 1

- SAT → k-Set Cover

Satisfiability Problem (SAT)

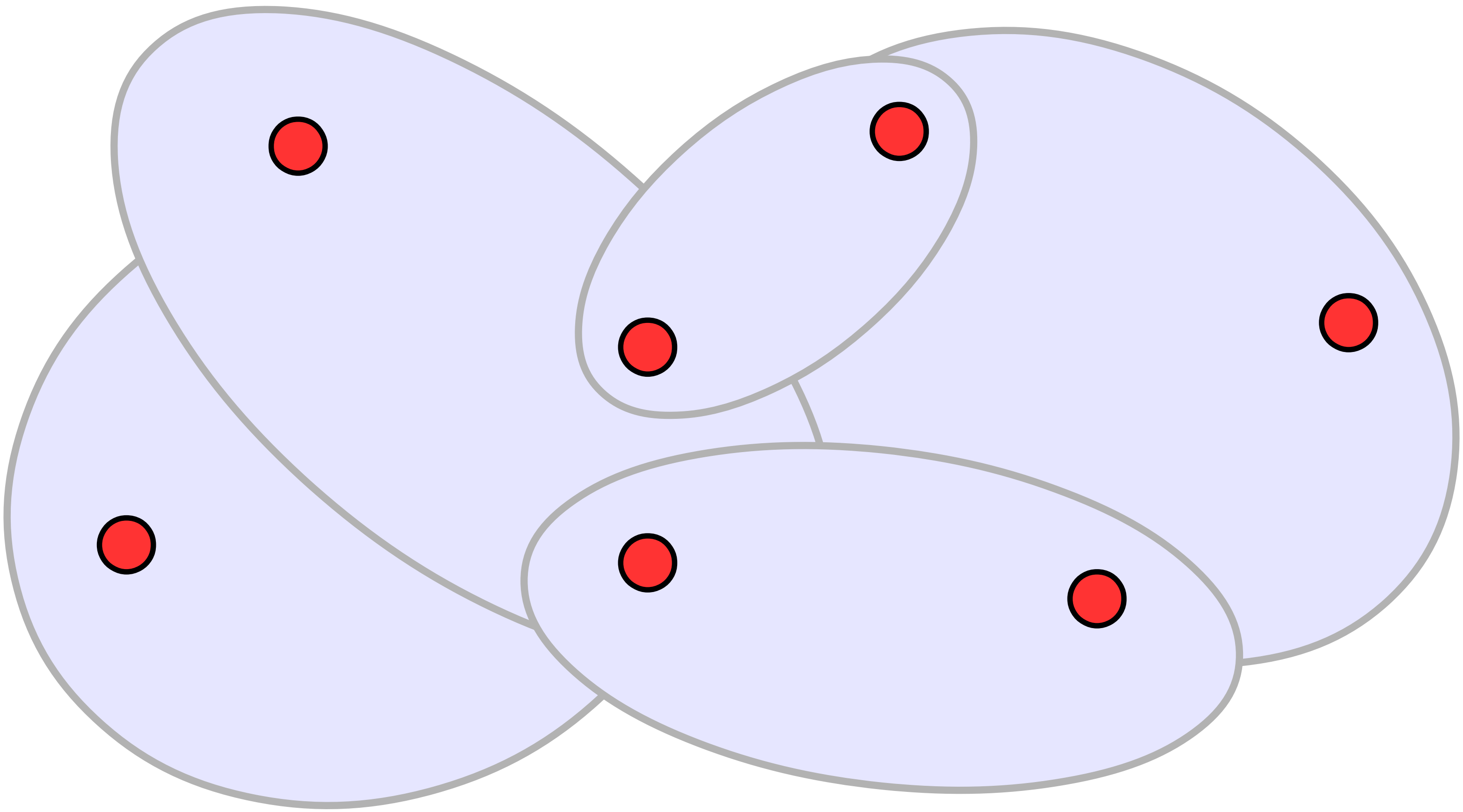
Given a CNF formula I (formula: (... or ...) and (... or ...) and (... or ...) .)

e.g., (x_1 or (**not** x_2) or x_3) and (x_2 or x_3 or (**not** x_4))

Goal: decide whether I is satisfiable.

k-Set Cover (k-SCP)

Given a collection of sets $S_1, \ldots, S_n$ and elements $e_1, \ldots, e_m$
Decide whether there are k sets that covers all the elements.

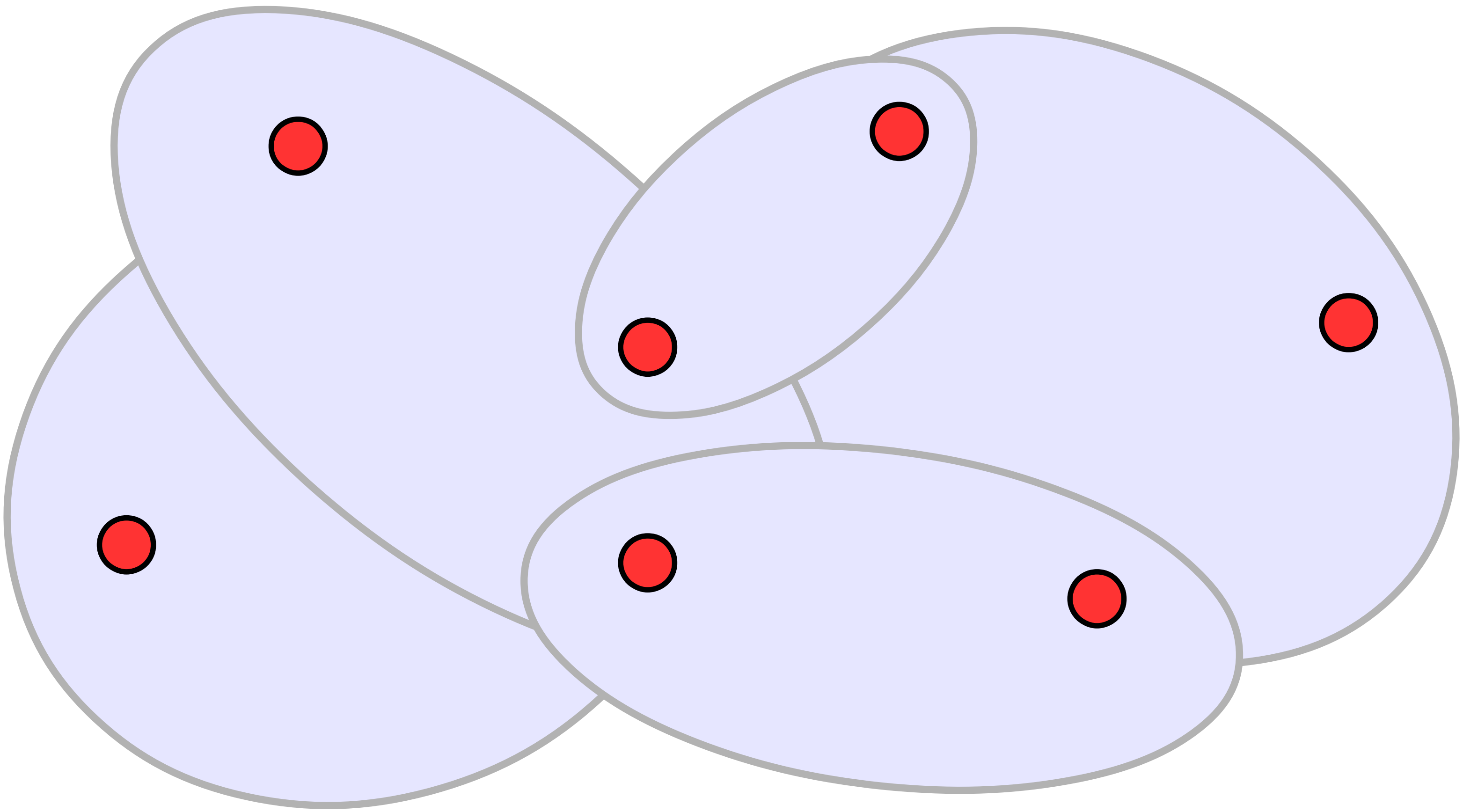


SAT → k-Set Cover

We want to show k-SCP cannot be solved in $n^{(0.99)k}$ time unless SAT can be solved in $2^{0.99n}$ time.

SAT → (n)-Set Cover

List all possible settings of each variables as sets.
List all clauses as elements.



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SAT → k-Set Cover

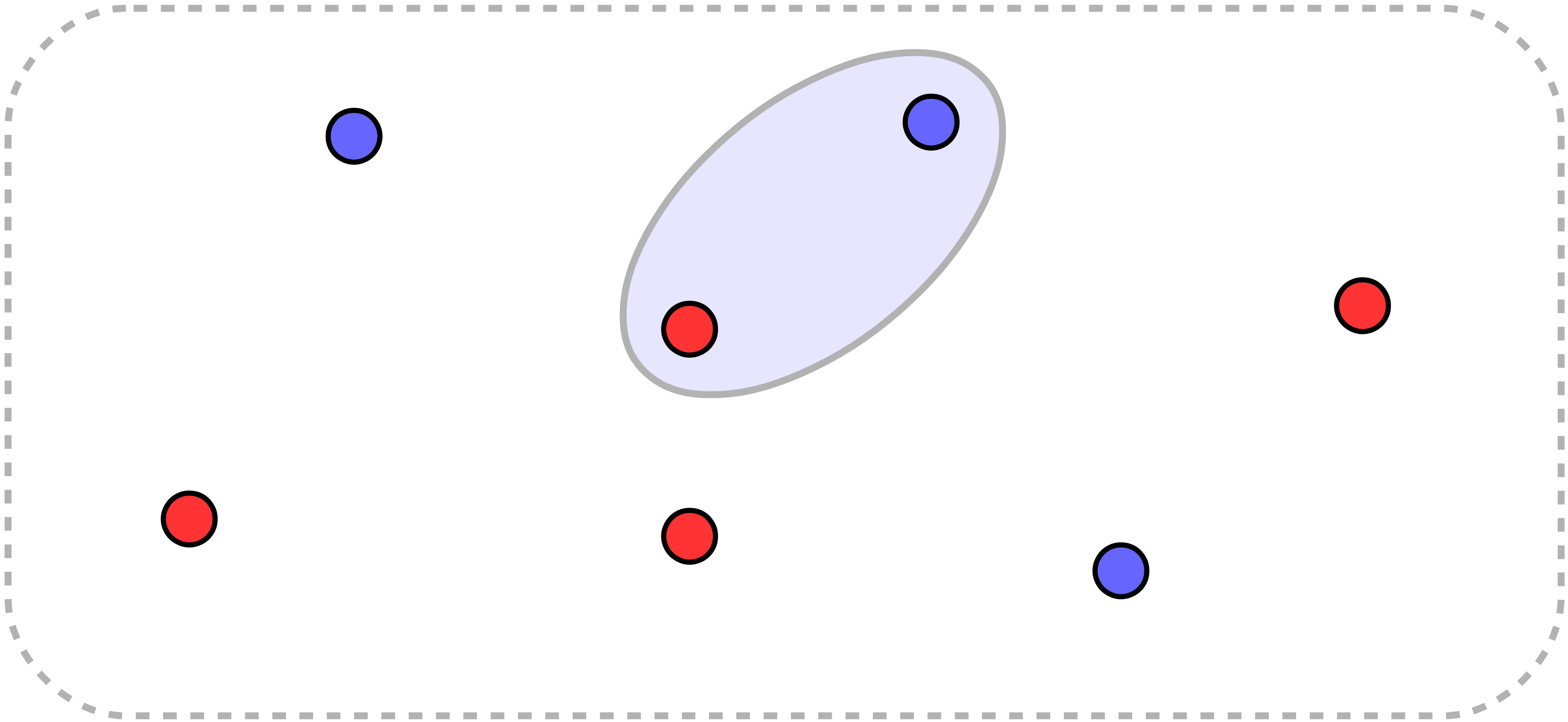
We want to show k-SCP cannot be solved in $n^{(0.99)k}$ time
unless SAT can be solved in $2^{0.99n}$ time.

Warming-Ups 2

- SAT → Bi-Chromatic Closest Pair

Bi-Chromatic Closest Pair (BCP)

Given a collection of n red and n blue points in a d-dimensional metric, find a pair of red-blue points with minimum-distance.



Satisfiability Problem (SAT)

Given a CNF formula I (formula: (... or ...) and (... or ...) and (... or ...) .)

e.g., (x_1 or (**not** x_2) or x_3) and (x_2 or x_3 or (**not** x_4))

Goal: decide whether I is satisfiable.

SAT → BCP

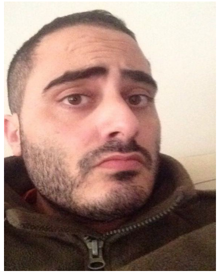
Goal: prove n^2 time lower bound for BCP.
Hint: Similar to SAT → 2-SCP

Second Part of the Talk

- Complexity of Closest Pair
- Geometric Representation of Graphs

Complexity of Closest Pair via Polar Pair of Point-Sets

Joint work with



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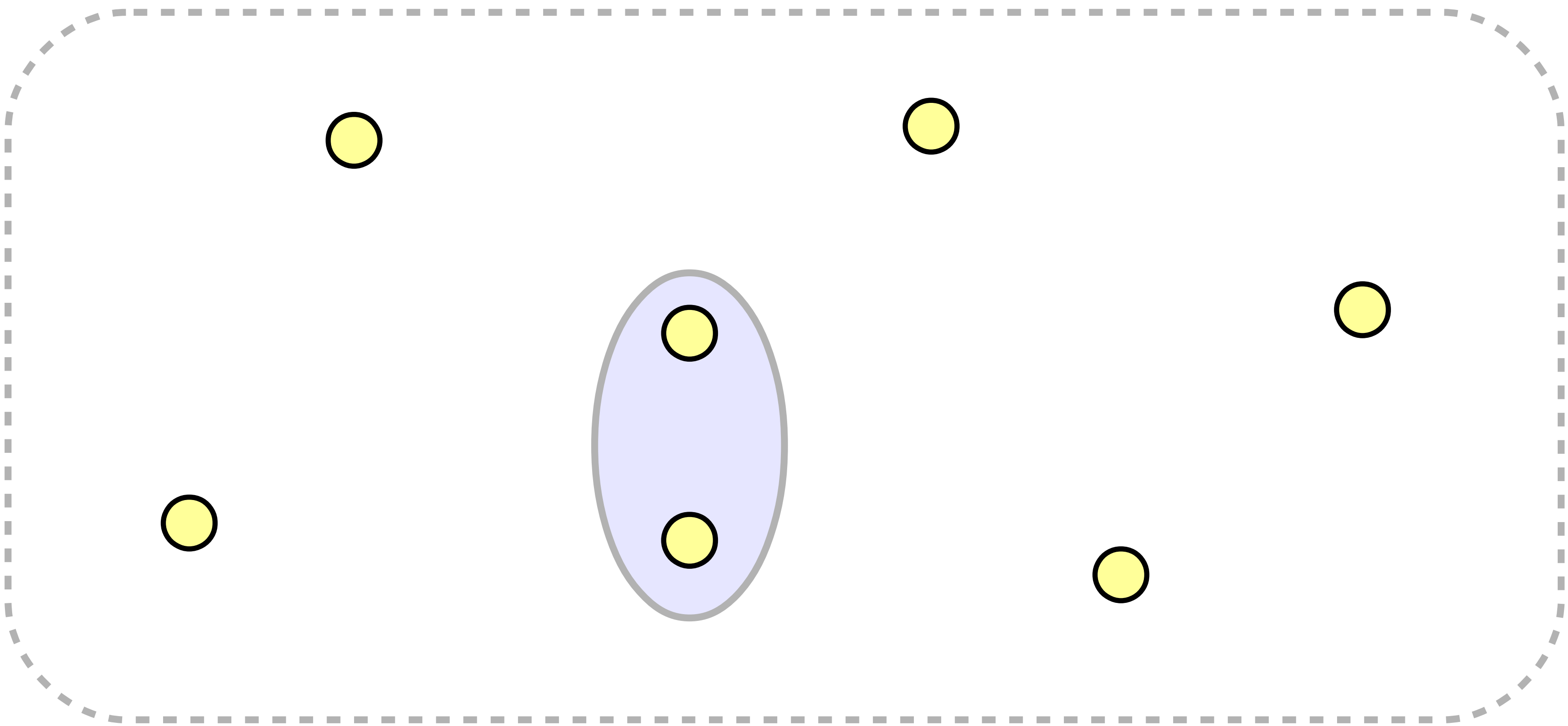
Weizmann Institute

Roee David
Datorama

Closest Pair (CP)

Given a collection of n points in a d-dimensional metric,
find a pair of points with minimum-distance.



Known Algorithms

- Euclidean Closest Pair
  - Dimension $d=O(1)$:
    - $O(2^D n \log n)$ (deterministic) [Bently-Shamos'76]
    - $O(2^D n)$ (randomzied) [Rabin'76, Khuller-Mattias'95]
  - Dimension $d=\Theta(\log n)$
    - $O(d n^2)$ (trivial algorithm)
  - Dimension $d=n$: $O(n^{3-\varepsilon})$, for some $\varepsilon > 0$

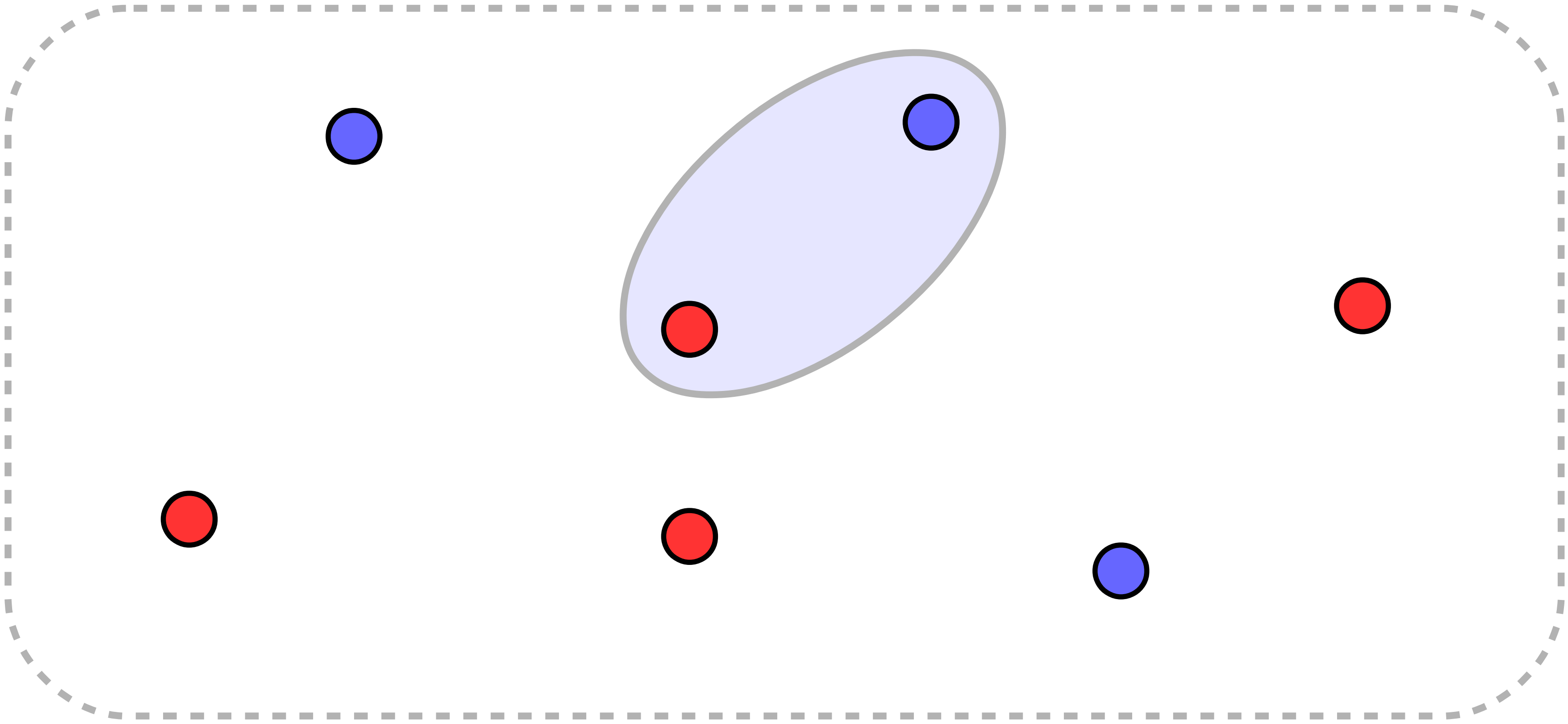
Is there an $O(n^{1.9})$-time algorithm when dimension $d=\Omega(\log n)$?

Is there an $O(n^{1.9})$-time algorithm when dimension $d=\Omega(\log n)$?

Don't know for Euclidean Closest Pair.
No for the bichromatic variant.

Bi-Chromatic Closest Pair (BCP)

Given a collection of n red and n blue points in a d-dimensional metric, find a pair of red-blue points with minimum-distance.



Bi-Chromatic Closest Pair
has no $O(n^{1.9})$-time algorithm
for any L^P-metric
unless SAT has $(2^{1.9n})$-time algorithm

Alman-Williams 2015

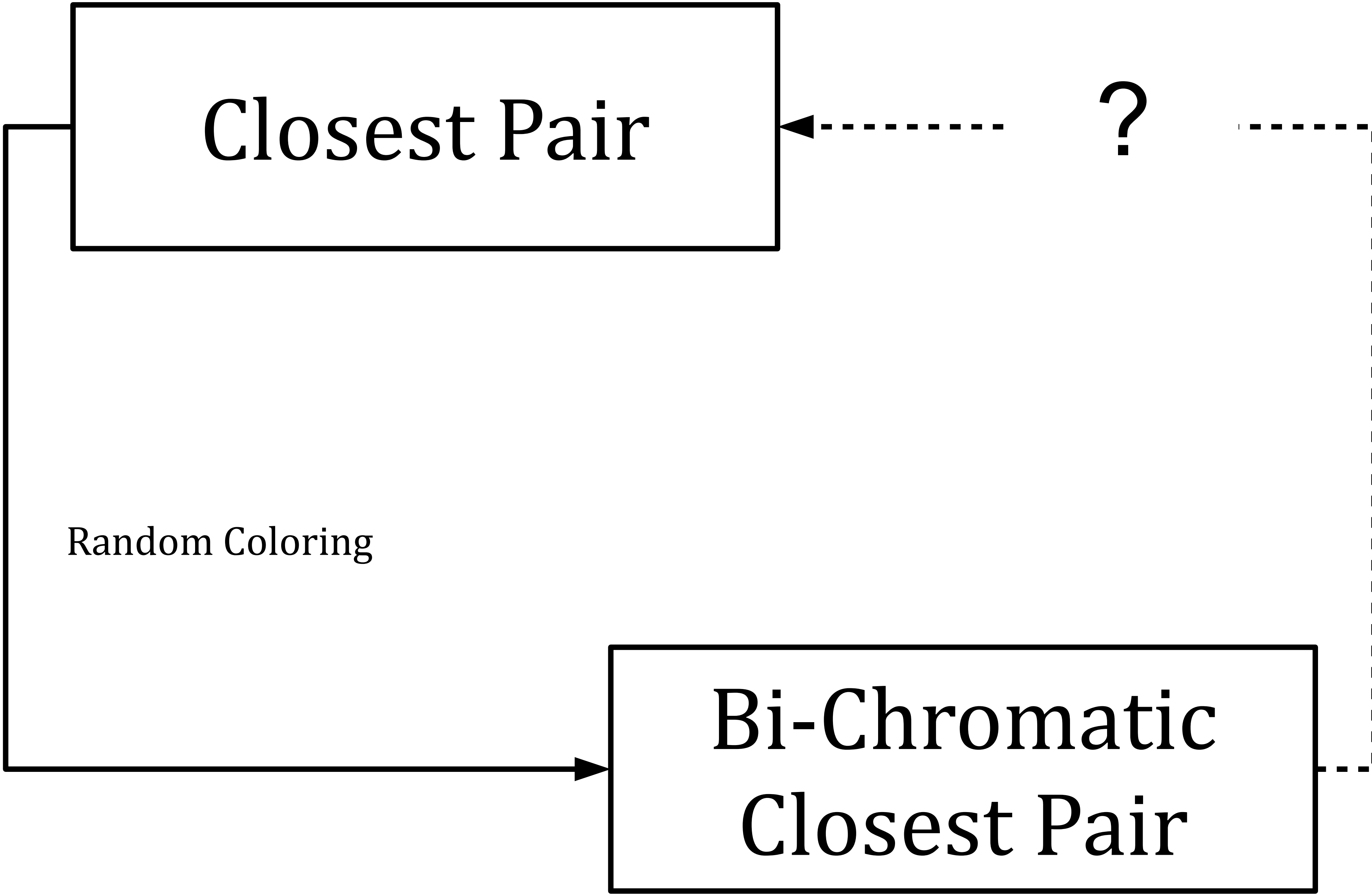
Bi-Chromatic Closest Pair has no $O(n^{1.9})$-time algorithm for any L^P-metric

unless SAT has $(2^{1.9n})$-time algorithm

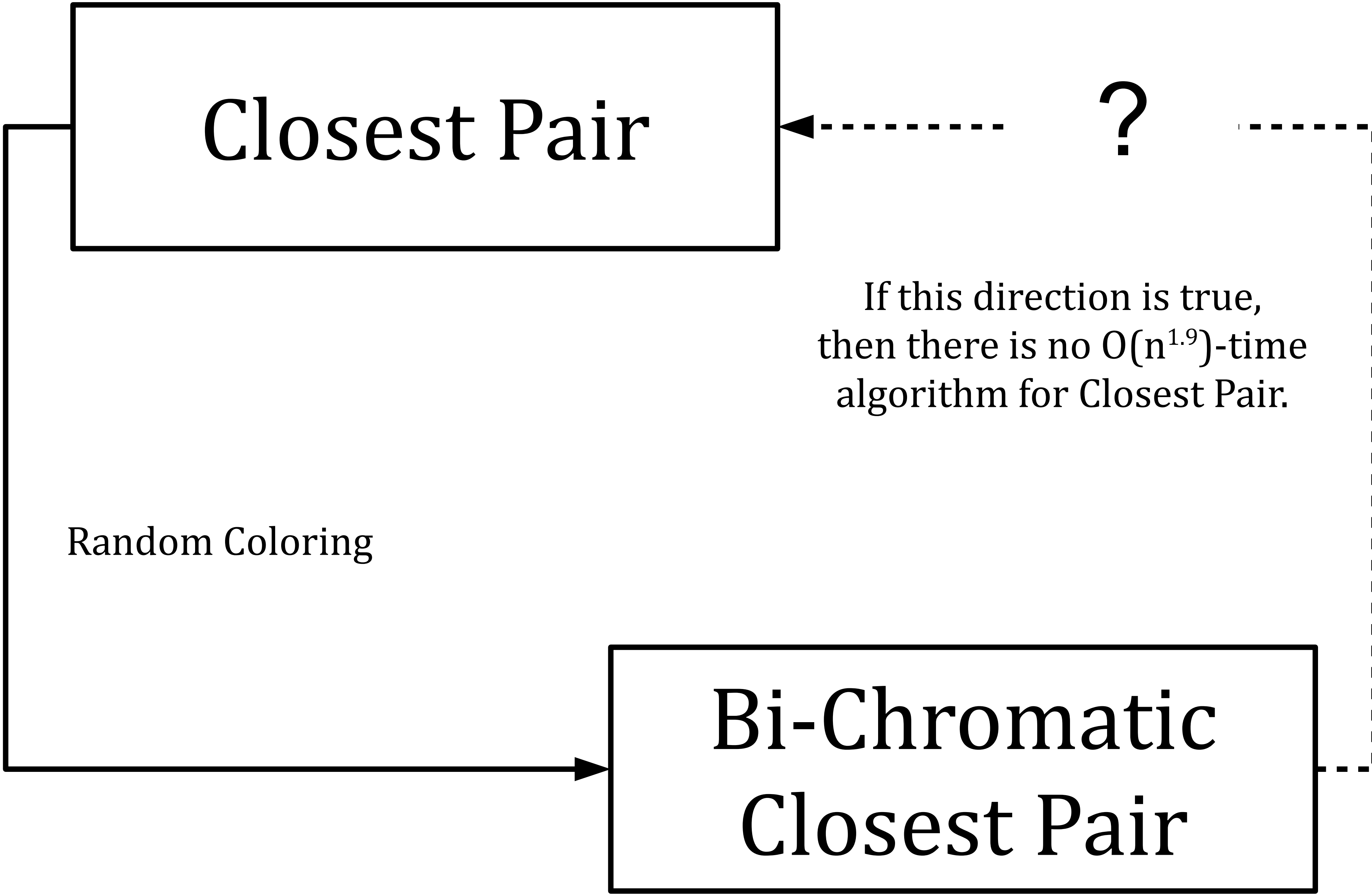
Alman-Williams 2015

Even $(1+o(1))$-approx in $O(n^{1.9})$-time is not possible.

Rubinstein 2018

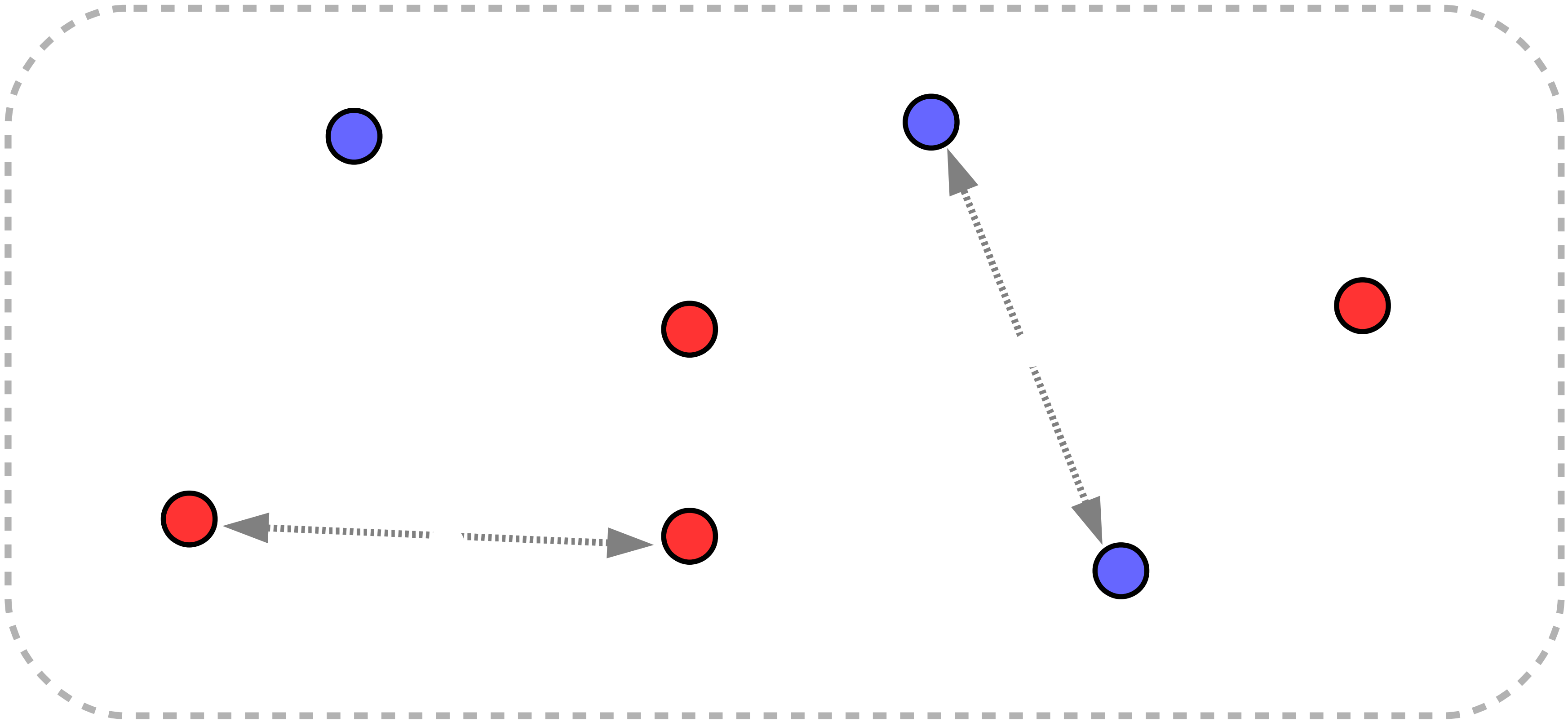


Closest Pair
?
Random Coloring
Bi-Chromatic
Closest Pair


Closest Pair
?
If this direction is true, then there is no $O(n^{1.9})$-time algorithm for Closest Pair.
Random Coloring
Bi-Chromatic Closest Pair

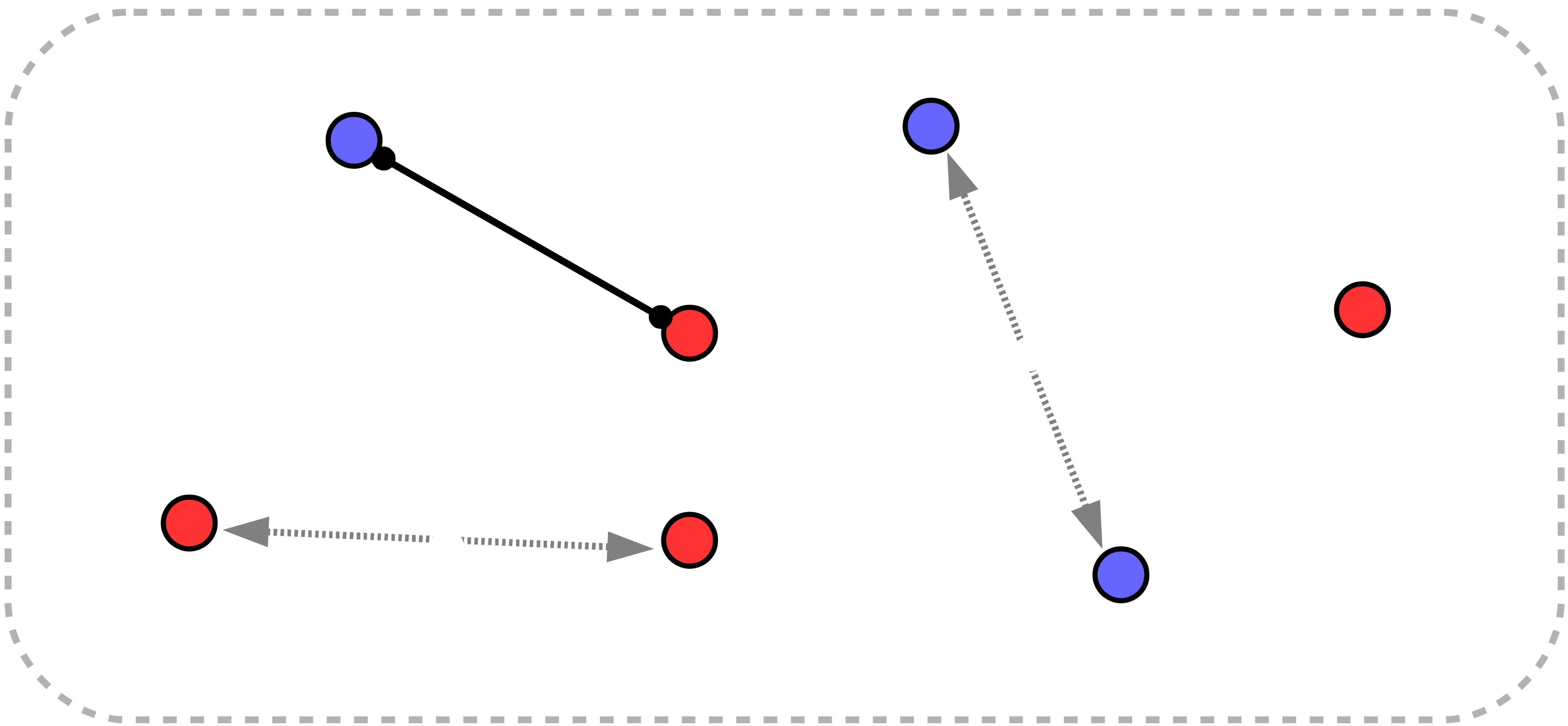
Reduction BCP → CP

Enlarge Red-Red, Blue-Blue distances.



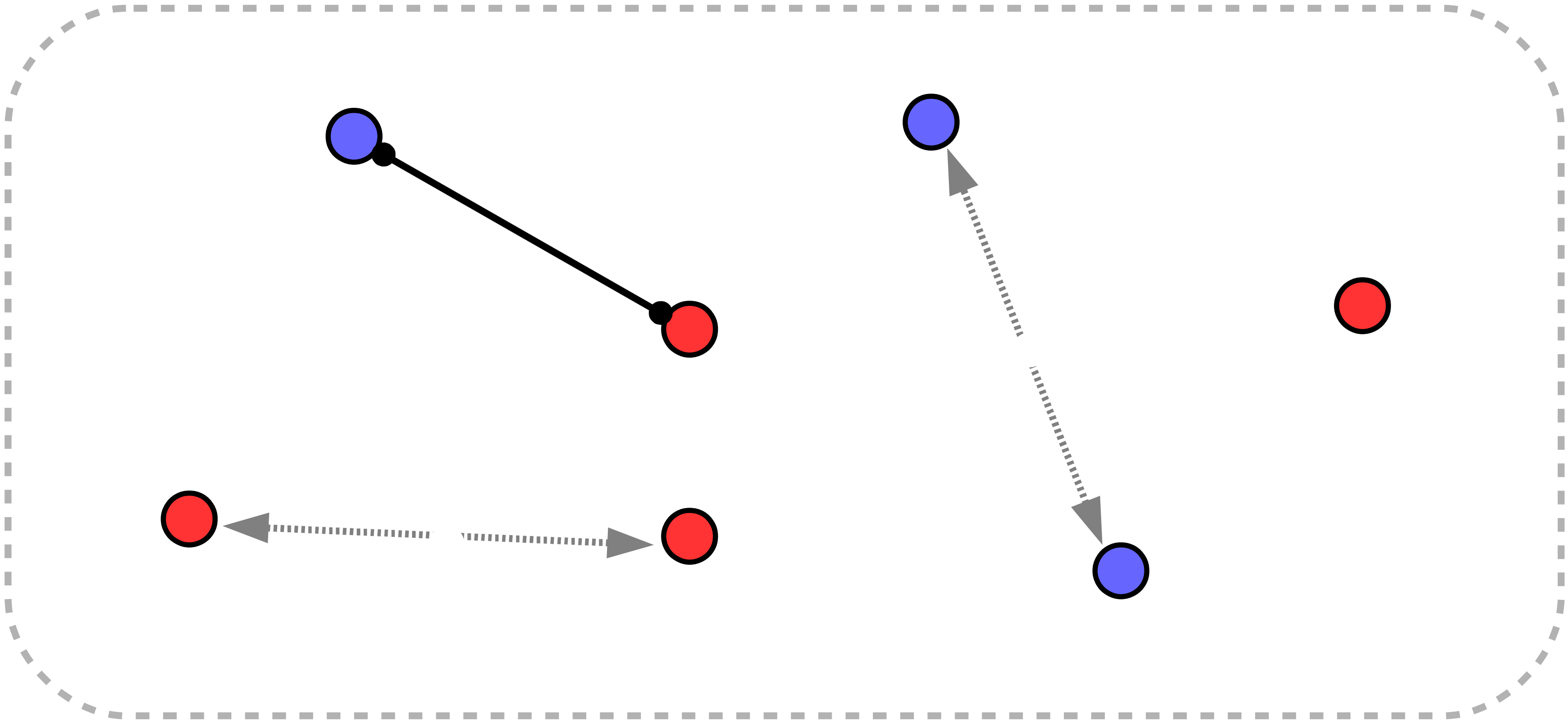
Reduction BCP → CP

Enlarge Red-Red, Blue-Blue distances.



Reduction BCP → CP

Concatenate point-vectors with codewords



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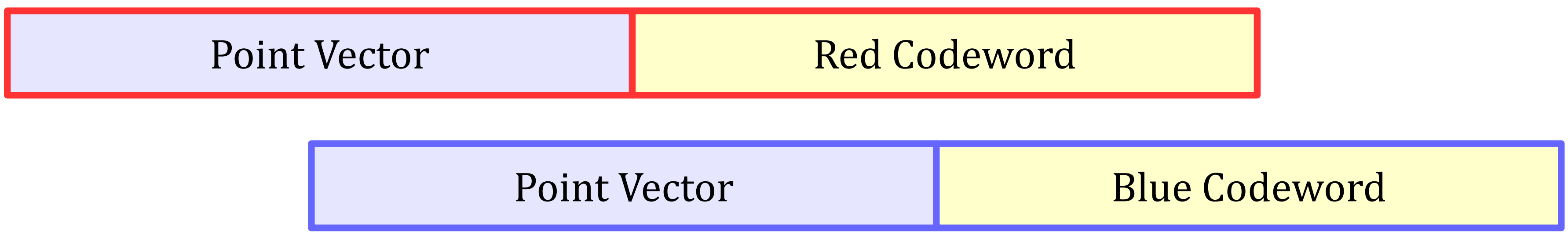

Reduction BCP → CP

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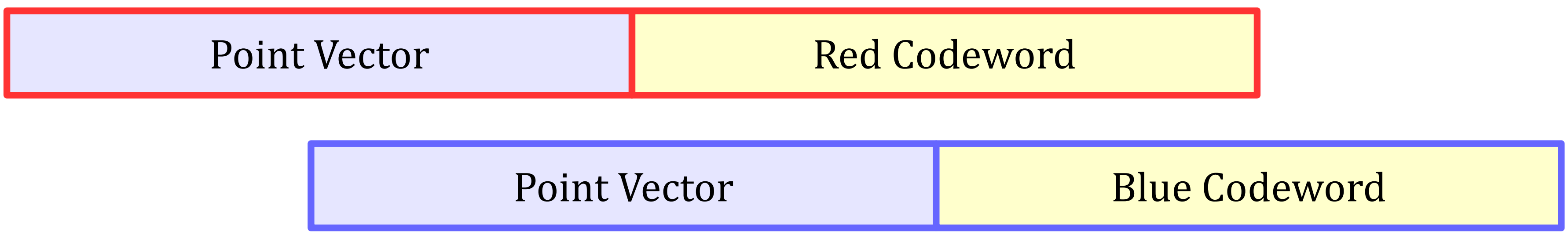


Needed Properties of The Codewords
(Bi-Clique Property)

Distance(Red-Code, Red-Code') ≥ R + 1/n
Distance(Blue-Code,Blue-Code') ≥ R + 1/n
Distance(Red-Code, Blue-Code) = R

The existence of Codewords with Bi-Clique Property implies BCP → CP

(that runs in $O(n^{1.9})$-time)

The existence of Codewords with Bi-Clique Property implies BCP → CP

(that runs in $O(n^{1.9})$-time)

Proof

Scale each point so that maximum distance $\leq 1/n$.

Concatenate each red/blue point with red/blue codewords.

For any (new) red-points $red = r_1 \cdot rc_1$ and $red' = r_2 \cdot rc_2$

$$dist(red, red') = (dist(r_1, r_2)^2 + dist(rc_1, rc_2)^2)^{1/2} > R + 1 / n$$

$$(\text{since } dist(\text{red-code}, \text{red-code}') > R + 1/n)$$

For any (new) red & blue point $red = r \cdot rc$ and $blue = b \cdot bc$

$$dist(red, blue)^2 = (dist(r, b)^2 + dist(rc, rb)^2)^{1/2}$$

$$= (dist(r, b)^2 + R^2)^{1/2} < (1/n + R^2)^{1/2} < dist(red, red')$$

⟹ Any red-blue closest-pair is the closest-pair in the new instance.

■

The existence of Codewords with Bi-Clique Property implies BCP → CP

(that runs in $O(n^{1.9})$-time)

Proof

Scale each point so that maximum distance ≤ 1/n.

Concatenate each red/blue point with red/blue codewords.

For any (new) red-points red = $r_1 \cdot rc_1$ and red' = $r_2 \cdot rc_2$

$$\text{dist}(\text{red}, \text{red}') = (\ \text{dist}(r_1, r_2)^2 + \text{dist}(rc_1, rc_2)^2\ )^{1/2} > R + 1/n$$

(since dist(red-code, red-code') > R + 1/n)

For any (new) red & blue point red = $r \cdot rc$ and blue = $b \cdot bc$

$$\text{dist}(\text{red}, \text{blue})^2 = (\ \text{dist}(r, b)^2 + \text{dist}(rc, rb)^2\ )^{1/2}$$

$$= (\text{dist}(r, b)^2 + R^2)^{1/2} < (1/n + R^2)^{1/2} < \text{dist}(\text{red}, \text{red}')$$

⟹ Any red-blue closest-pair is the closest-pair in the new instance.

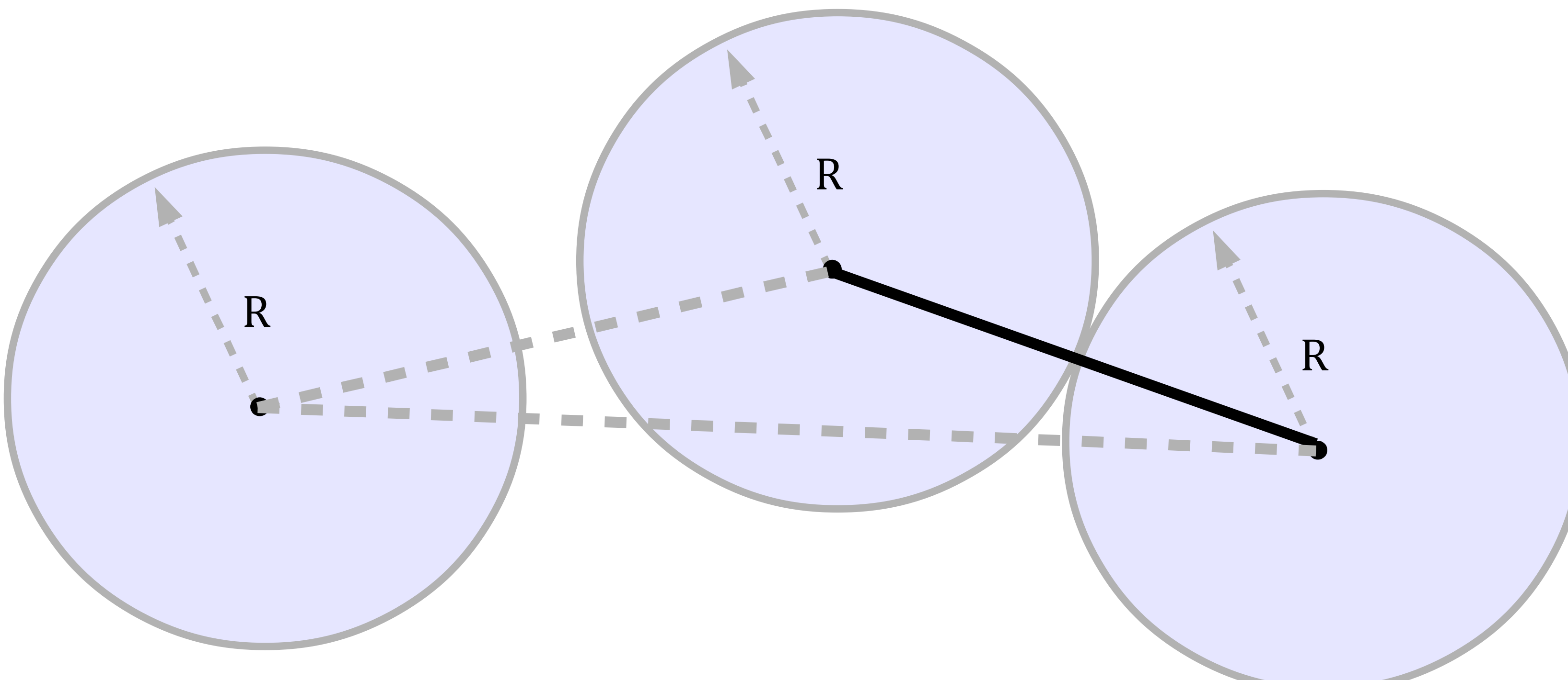
■

The existence of Codewords with Bi-Clique Property implies BCP → CP

(that runs in $O(n^{1.9})$-time)

Requires codewords construction with running time $O(n^{1.9})$ & dimension $O(\log n)$

Complexity Question of CP reduces to Geometric Representation of Bi-Clique

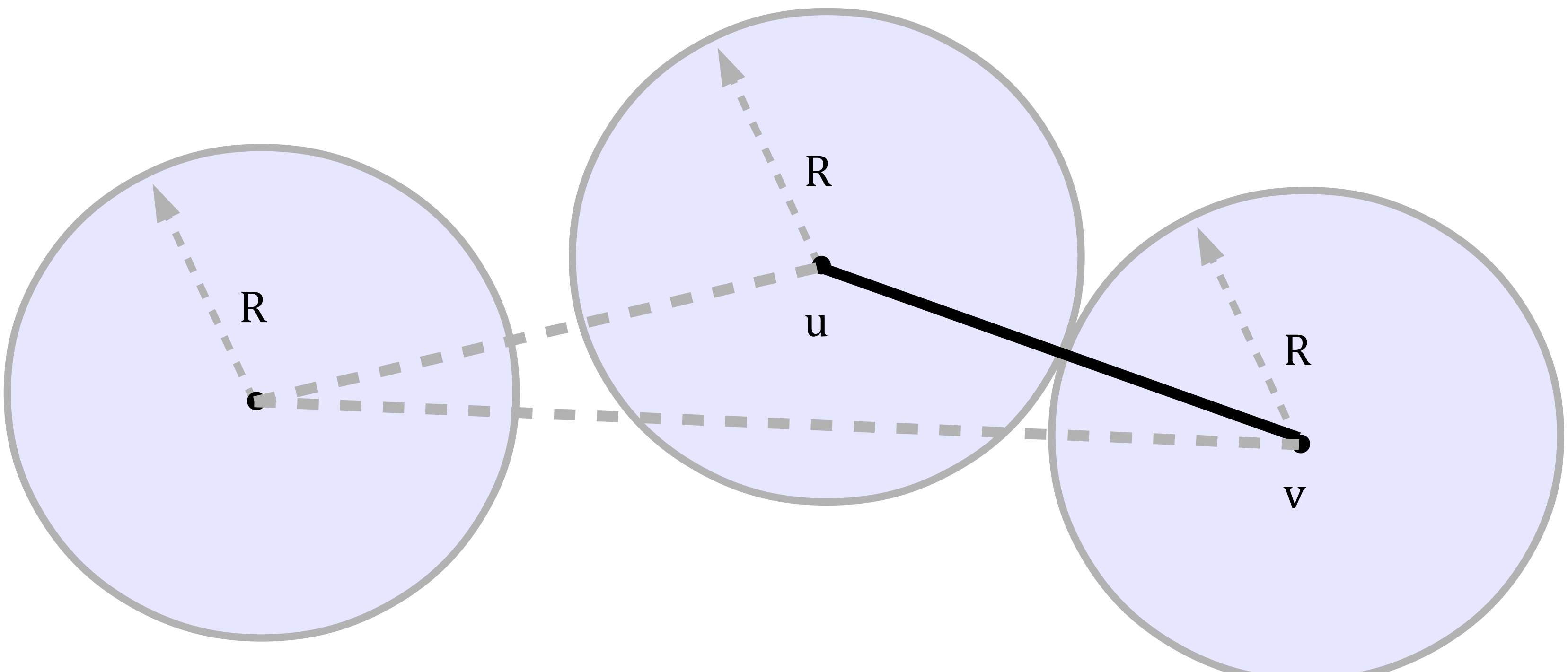


Complexity Question of CP reduces to Geometric Representation of Bi-Clique

What is the smallest dimension to represent a bi-clique in L^p-metric?

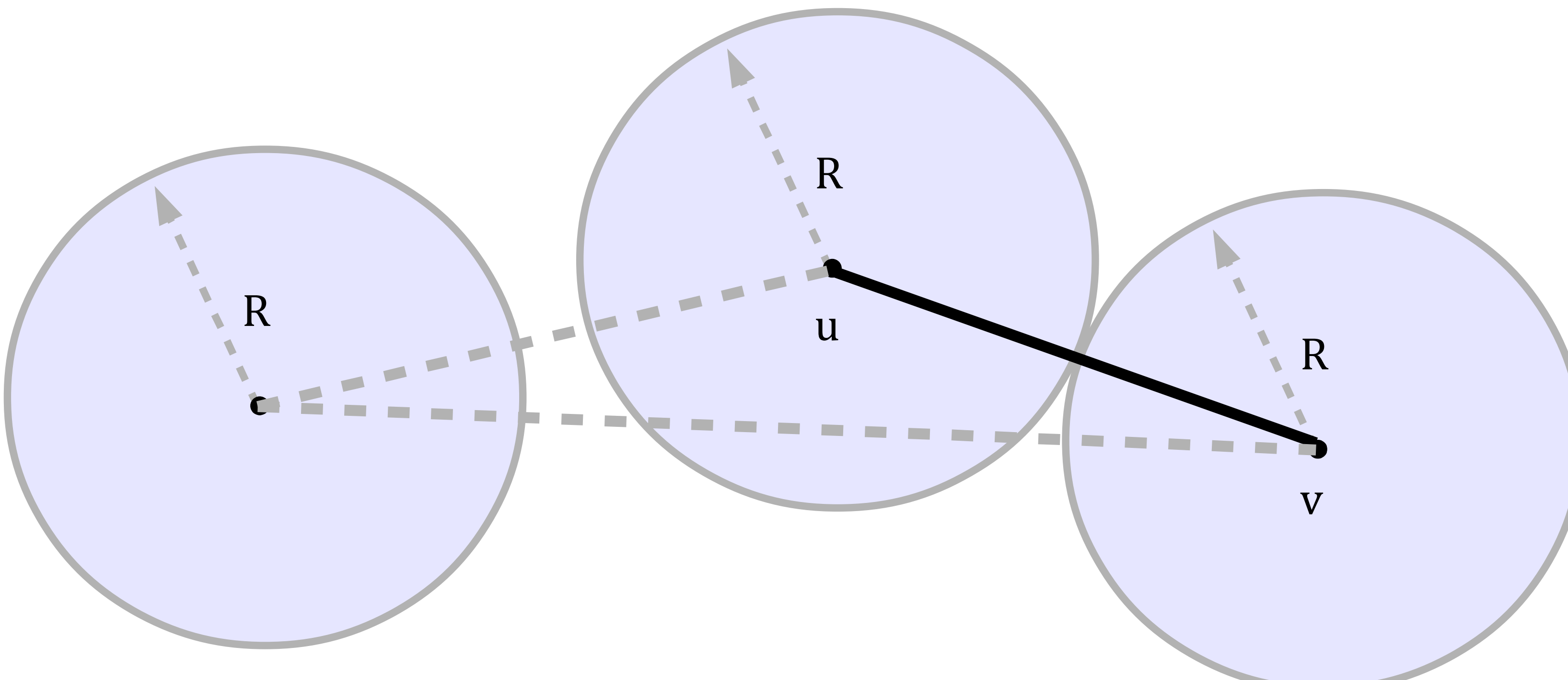
Contact-Graph

Vertices are non-overlapping d-dimensional spheres of radius R.
There is an edge uv if Sphere(u) touches Sphere(v).



Contact-Dimension of Bi-Clique in L^p

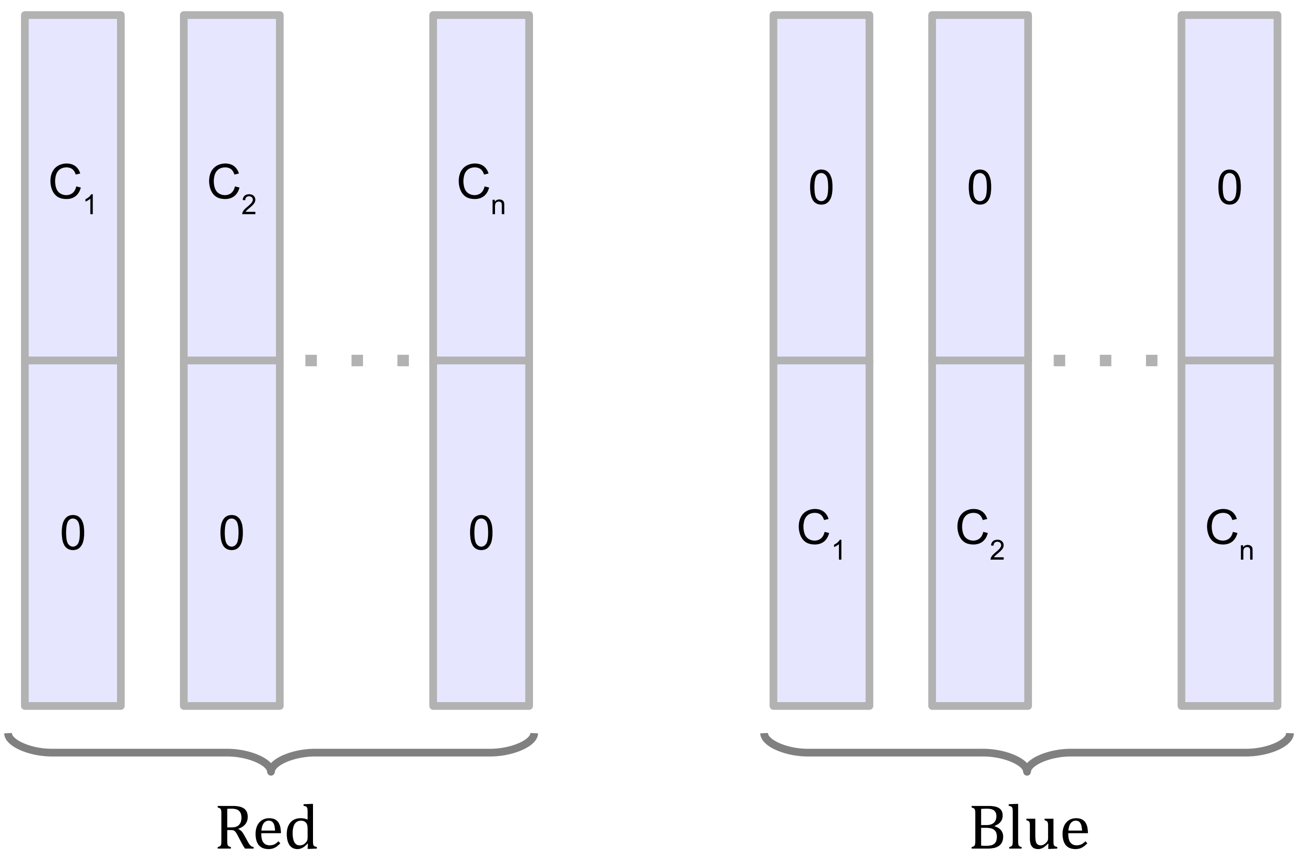
bicd(L^p) = the smallest d such that $K_{n,n}$ can be represented by a contact graph in the d-dimensional L^p metric.



bicd(L^p) ≤ O(log n) for p > 2

Concatenate random codewords C in $\{1,-1\}^n$ with 0^n

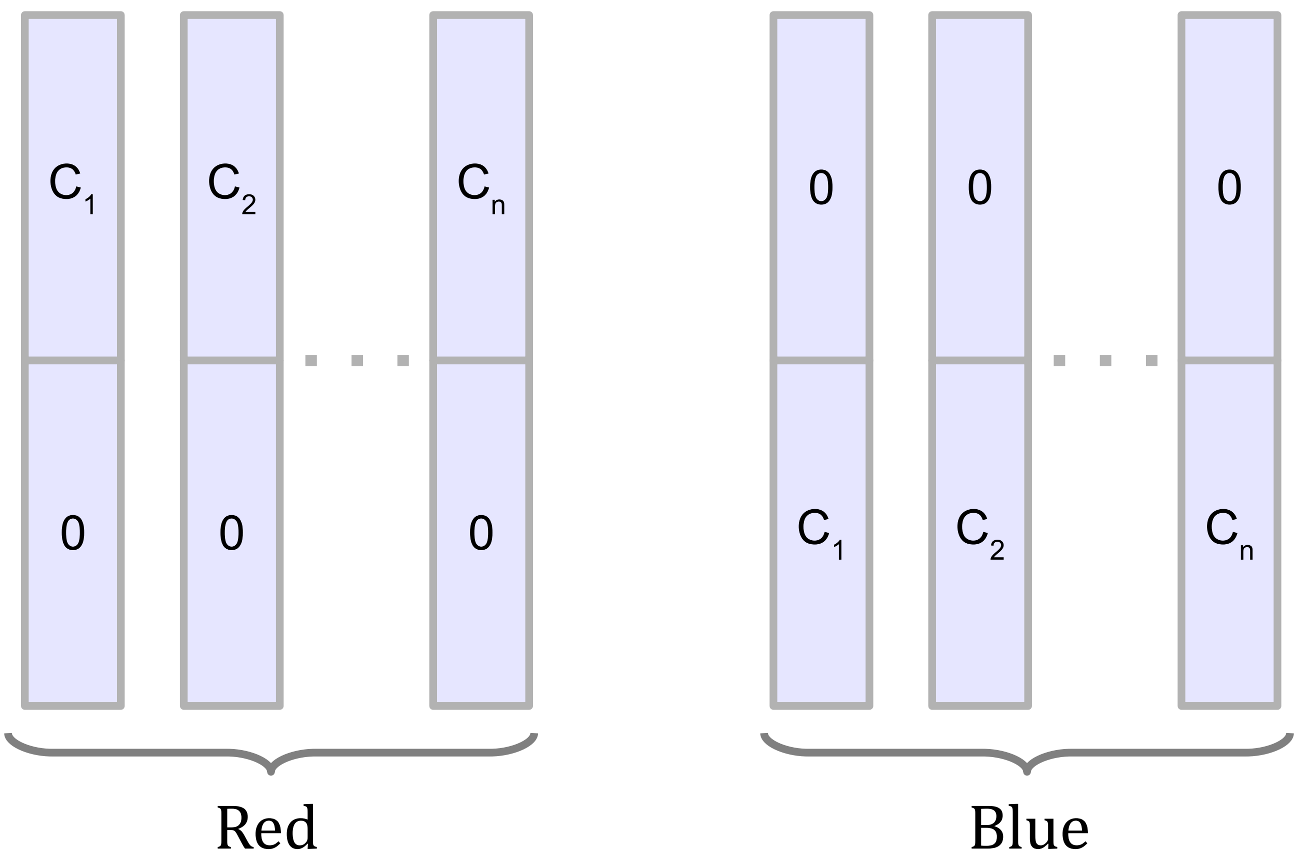
Red-codeword = C · 0^n, Blue-codeword = 0^n · C



$\mathrm{bicd}(L^p) \le O(\log n)$ for $p > 2$

Concatenate random codewords C in $\{1,-1\}^n$ with 0^n

Red-codeword = $C \cdot 0^n$, Blue-codeword = $0^n \cdot C$



$\mathrm{dist}(\text{red}, \text{blue})^p = 2n$

$\mathrm{dist}(\text{red}, \text{red}')^p$

$> (\frac{1}{2} - \varepsilon)\, n \cdot 2^p$

$> 2n$

(with high probability)

CP & BCP are equivalent in L^p-metrics.

(under subquadratic-time reduction)

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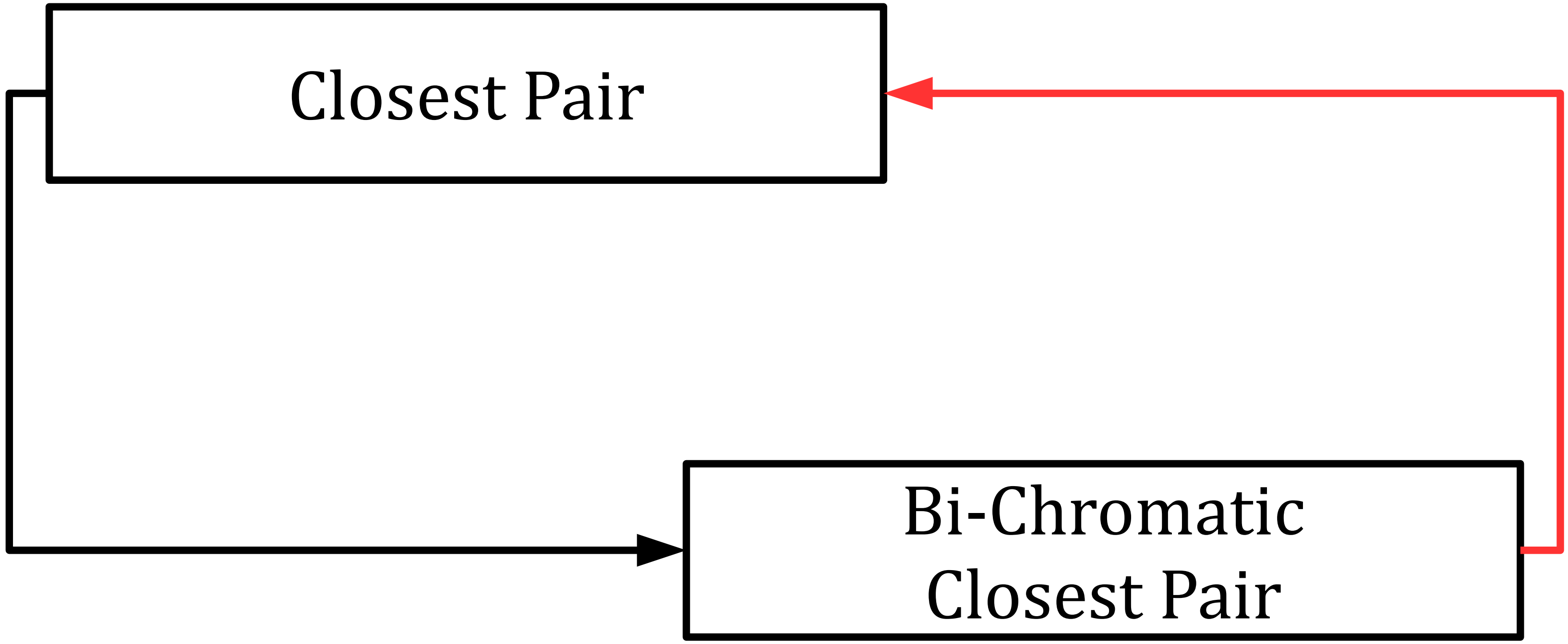

How about other L^p-metrics?

How about other L^p-metrics?

Yes for L^∞, **No** for L^0 and L^2, **Don't know** for L^1

$$1.286\, n \leq \mathrm{bicd}(L^2) \leq 1.5\, n$$

Maehara 1985, Frankl-Maehara 1988

$$1.286\, n \le \mathrm{bicd}(L^2) \le 1.5\, n$$

Maehara 1985, Frankl-Maehara 1988

Biclique has no contact-graph
in $O(\log n)$-dimensional L^2-metric.

$$n \leq \mathrm{bicd}(L^0) \leq n$$ (hamming metric)

$$? \leq \mathrm{bicd}(L^1) \leq n^2$$

$$? \leq \mathrm{bicd}(L^p) \leq n$$ for $1 < p < 2$

$$\Omega(\log n) \leq \mathrm{bicd}(L^\infty) \leq 2 \log_2 n$$

David, Karthik CS, **L.** 2018

$$n \le \mathrm{bicd}(L^0) \le n \quad \text{(hamming metric)}$$

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David, Karthik CS, **L.** 2018

Biclique has no contact-graph
in O(log n)-dimensional L^0-metric.

Current Bounds

for Biclique Contact Dimension

n	$\leq \text{bicd}(L^0)$	$\leq n$	
?	$\leq \text{bicd}(L^1)$	$\leq n^2$	
?	$\leq \text{bicd}(L^p)$	$\leq n$	for $1 < p < 2$
$1.286\,n$	$\leq \text{bicd}(L^2)$	$\leq 1.5\,n$	
$\Omega(\log n)$	$\leq \text{bicd}(L^p)$	$\leq O(\log n)$	for $p > 2$
$\Omega(\log n)$	$\leq \text{bicd}(L^\infty)$	$\leq 2 \log_2 n$	

Open Problems

- Better Lower / Upper Bounds for L^1 and L^2 ?
  - An alternative way to reduce BCP → CP ?

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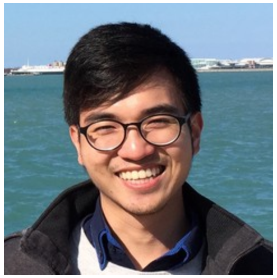
- Better Lower / Upper Bounds for L^1 and L^2 ?
  - An alternative way to reduce BCP → CP ?

Open problem solved:

Bichromatic & Classical Closest Pair are equivalent.

Recent work of



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UC Berkeley

The End

Thank you for your attention.

Questions?

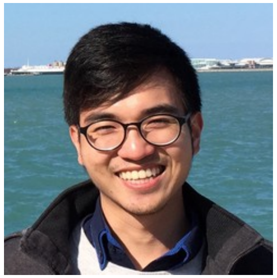
Third Part of the Talk

(extended part)

- Getting Running-Time Lower bound for Approximation Algorithms

On the Parameterized Complexity of Approximating Dominating Set

Joint work with



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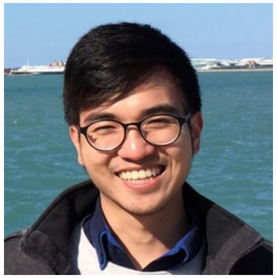
On the Parameterized Complexity of Approximating Dominating Set

Equivalent to Set Cover

Joint work with



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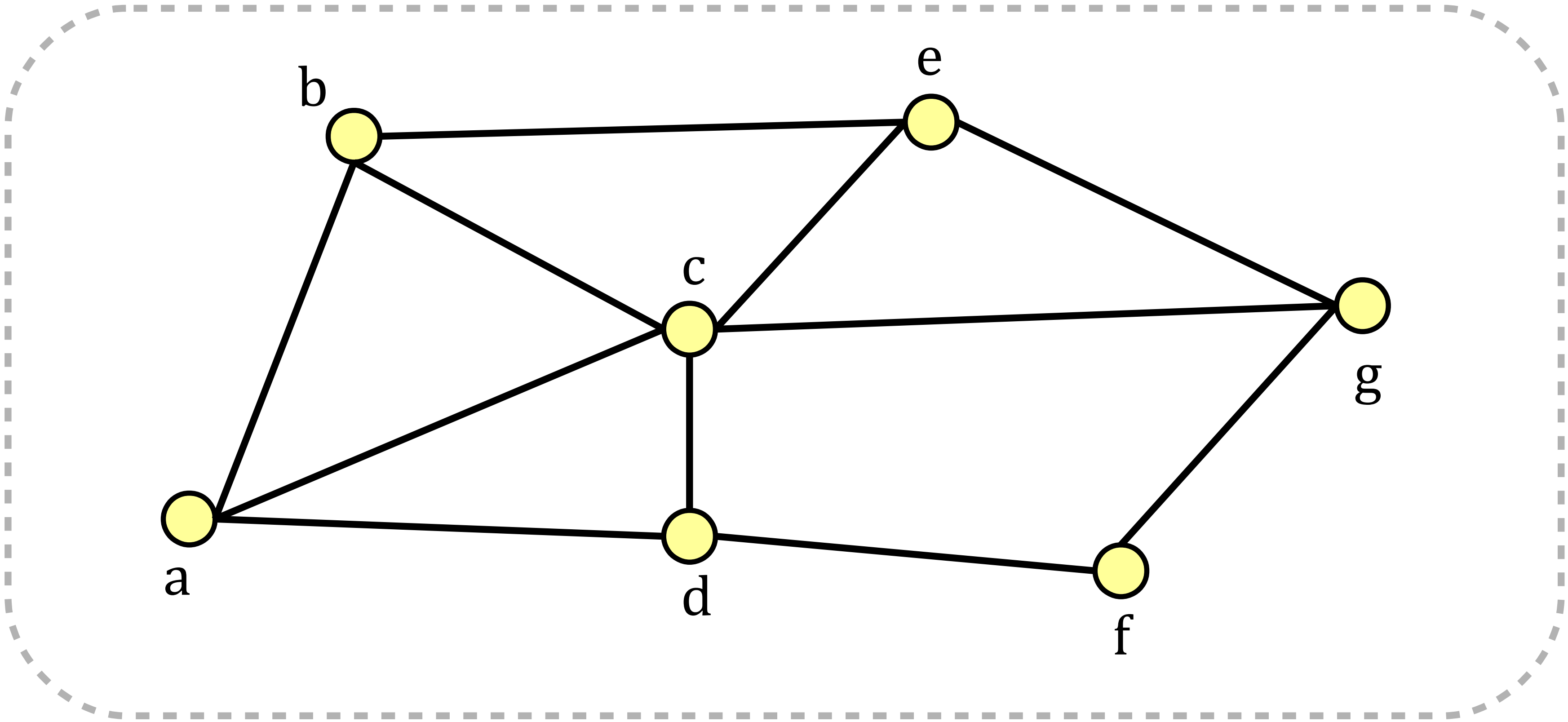
k-Dominating Set (k-DS)

Input: A graph G=(V, E) and an integer k > 0.
Goal: Find k vertices that dominates all other vertices.



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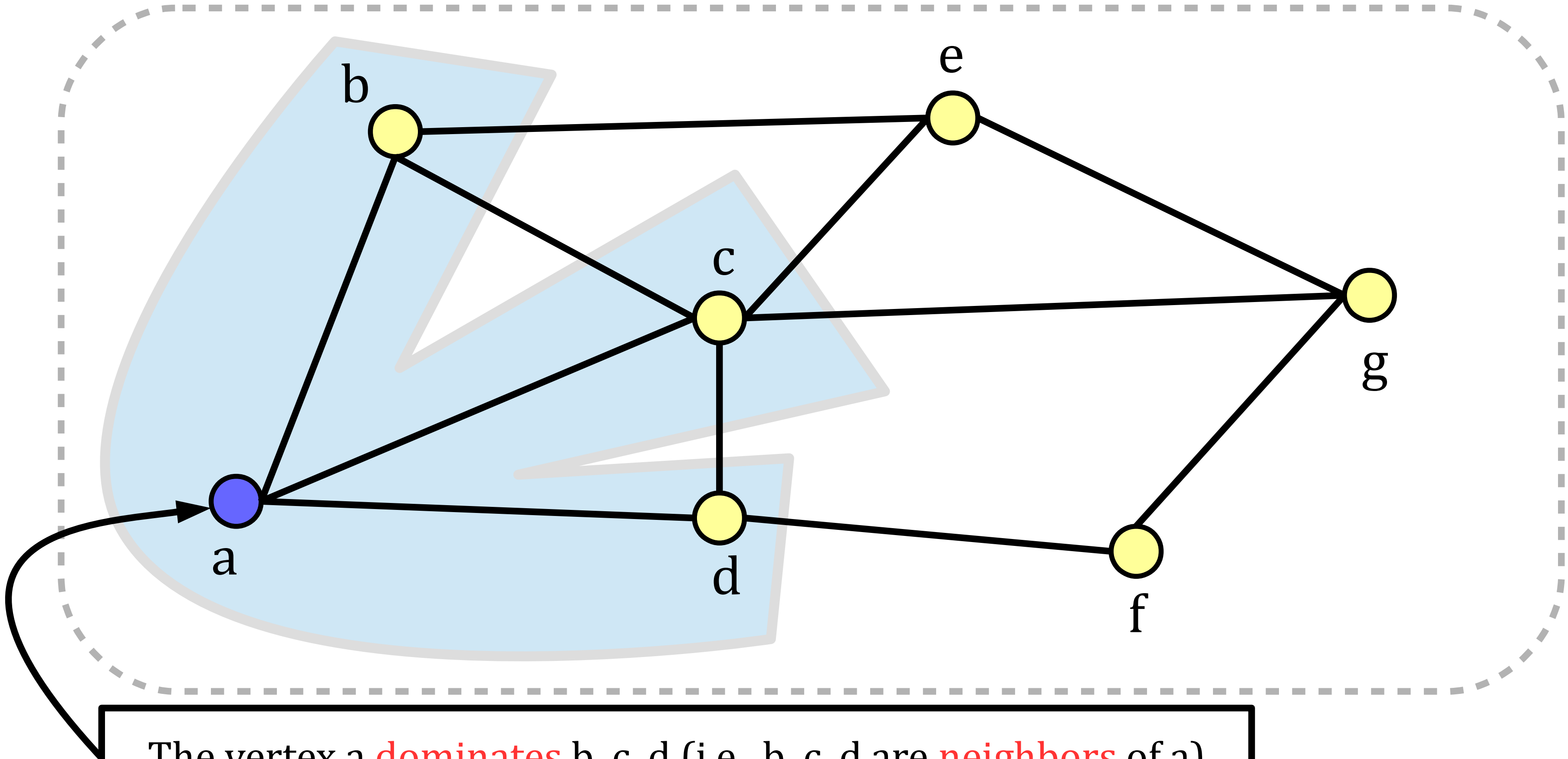

The vertex a dominates b, c, d (i.e., b, c, d are neighbors of a)

k-Dominating Set (k-DS)

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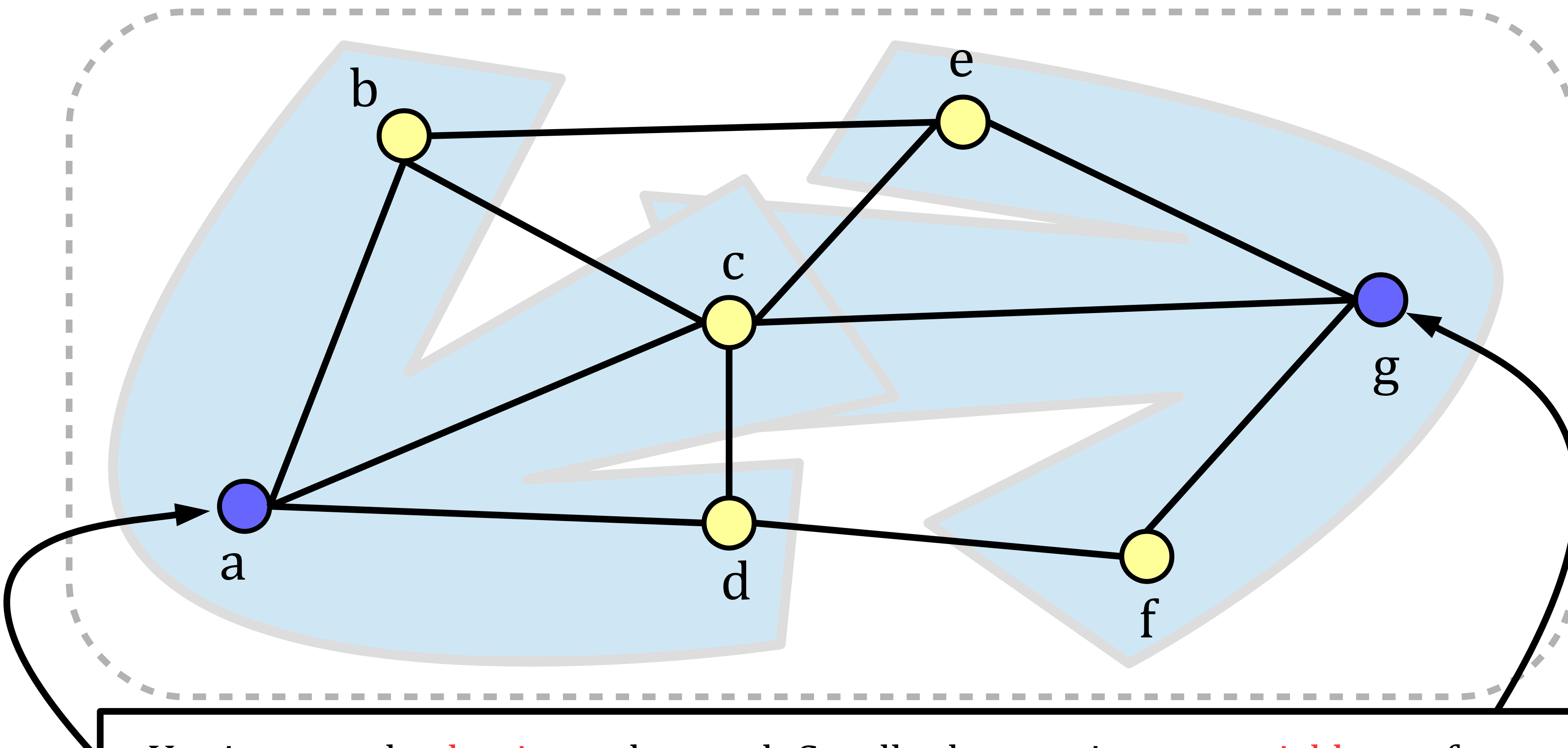

Vertices a and g dominate the graph G – all other vertices are neighbors of a or g

k-Dominating Set (k-DS)

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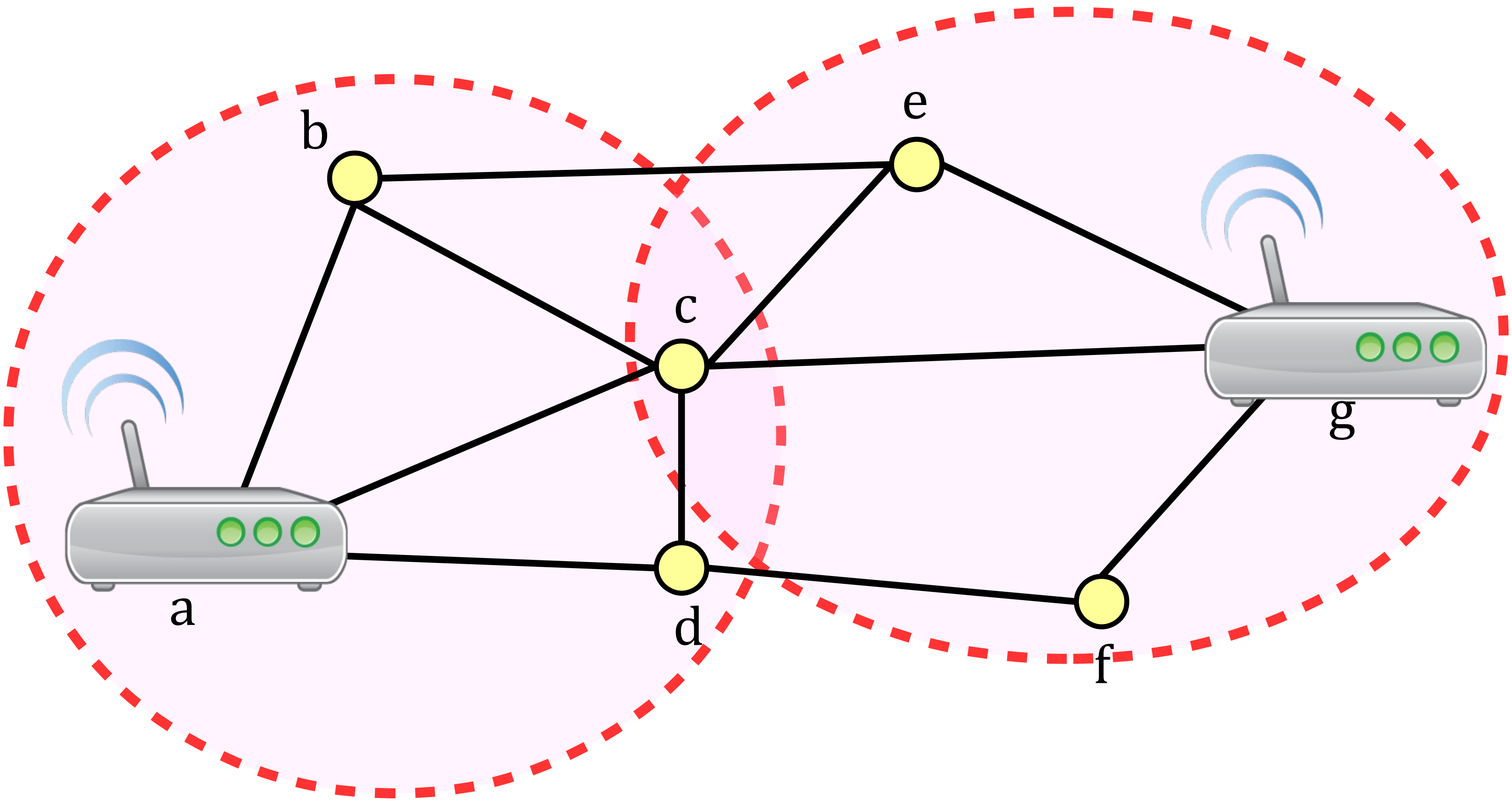

The Importance of Dominating Set

- W[2]-Complete Problem

Analogue of SAT in Parameterized Complexity

- Fine-Grained Lower Bound

Conjecture: No n^{k-1}-time algorithm for k-DS

The Importance of Dominating Set

- W[2]-Complete Problem

Analogue of SAT in Parameterized Complexity

- Fine-Grained Lower Bound

Conjecture: No n^{k-1}-time algorithm for k-DS

Inapproximability k-DS implies running-time bound of approximating many problems.

Known Algorithms

- Exact $n^{k+o(1)}$-time [PW'10]
- $O(\log n)$-approx in $O(n^3)$-time [J74]
- $(\ln n - \ln \ln n - \Theta(1))$-approx in $n^{O(1)}$-time [S96]

Known Algorithms

- Exact $n^{k+o(1)}$-time [PW'10]
- $O(\log n)$-approx in $O(n^3)$-time [J74]
- $(\ln n - \ln \ln n - \Theta(1))$-approx in $n^{O(1)}$-time [S96]

We don't know how to find a dominating set of size 2^k in time n^{k-1} even if we know that the graph has dominating set of size $k \ll n$.

Is there an $O(n^{k-1})$-time algorithm that gives $O(2^k)$-approximation?

Is there an $O(n^{k-1})$-time algorithm that gives $O(2^k)$-approximation?

Our answer is NO, and it is even WORSE!

(assuming popular hypotheses)

Our Results

Parameterized Inapproximability of k-Dominating Set

Hypothesis	Inapprox	Running Time	
$W[1] \neq FPT$	$\Omega(\log^{1/poly(k)} n)$	FPT-Time	(T(k) poly(n))
ETH	$\Omega(\log^{1/poly(k)} n)$	$n^{o(k)}$	
SETH	$\Omega(\log^{1/f(k)} n)$	$n^{k-\varepsilon}$	for any $\varepsilon > 0$
k-SUM	$\Omega(\log^{1/poly(k)} n)$	$n^{k/2-\varepsilon}$	for any $\varepsilon > 0$

Our Results

Parameterized Inapproximability of k-Dominating Set

Hypothesis	Inapprox	Running Time	
W[1]≠FPT	$\Omega(\log^{1/\text{poly}(k)} n)$	FPT-Time (T(k) poly(n))	
ETH	$\Omega(\log^{1/\text{poly}(k)} n)$	$n^{o(k)}$	
SETH	$\Omega(\log^{1/f(k)} n)$	$n^{k-\varepsilon}$	for any $\varepsilon > 0$
k-SUM	$\Omega(\log^{1/\text{poly}(k)} n)$	$n^{k/2-\varepsilon}$	for any $\varepsilon > 0$

Even exp(exp(exp(k)))-Approx in $(n^{k-0.1})$-Time is unlikely!

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Hardness of these forms (i.e., $\Omega(\log^{1/k} n)$-inapprox in $n^{o(k)}$-time) were known only under Gap-ETH. [CCK**L**MNT, FOCS'17]

Our Techniques

Parameterized Inapproximability via Communication Complexity

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We extend their works to the area of Parameterized Inapproximability.

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Parameterized Inapproximability via Communication Complexity

Our results follow by the connection between Communication Protocols & Label Cover.

(Precisely, the variant of Label Cover defined in [CCK**L**MNT'17], namely Max-Cover.)

We devise communication protocols for problems corresponding to the popular hypotheses FPT≠W[1], ETH, SETH, k-SUM.

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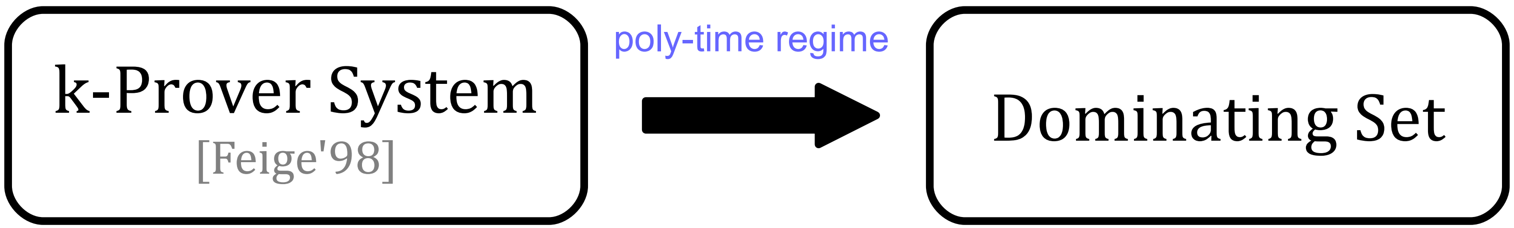
We extend their works to the area of Parameterized Inapproximability.

k-DS & Communication Protocols



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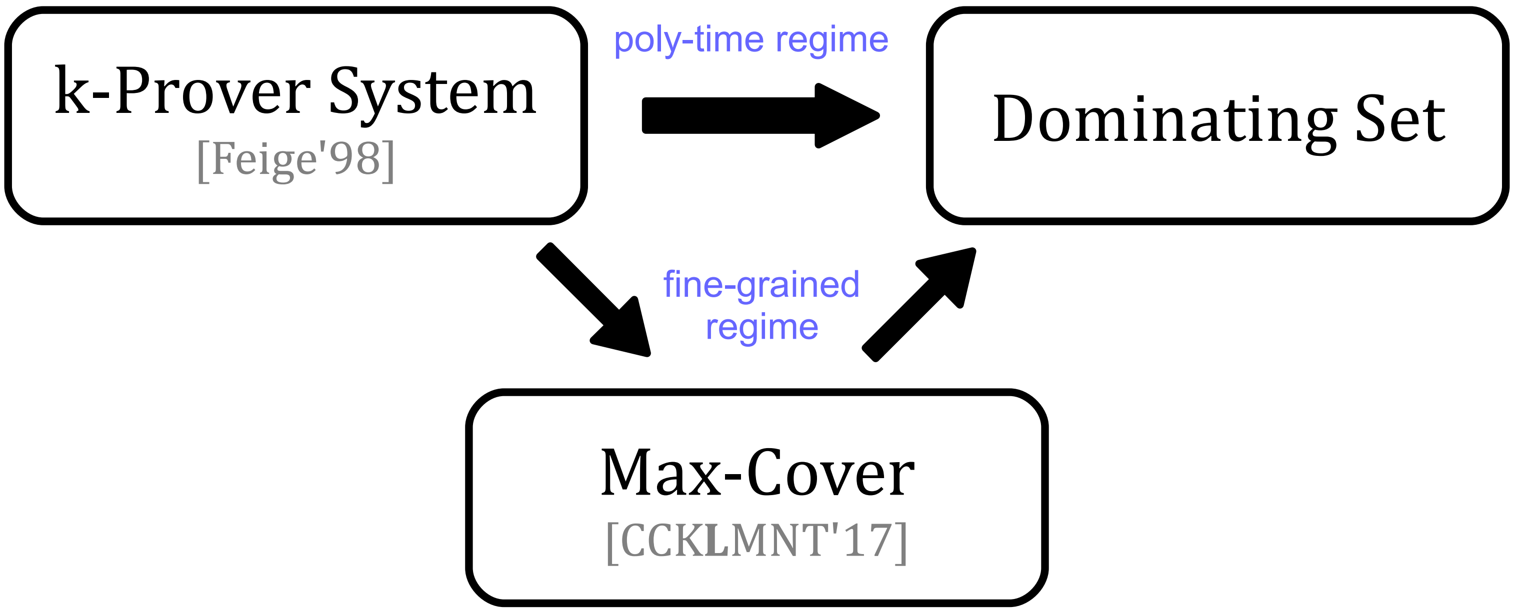

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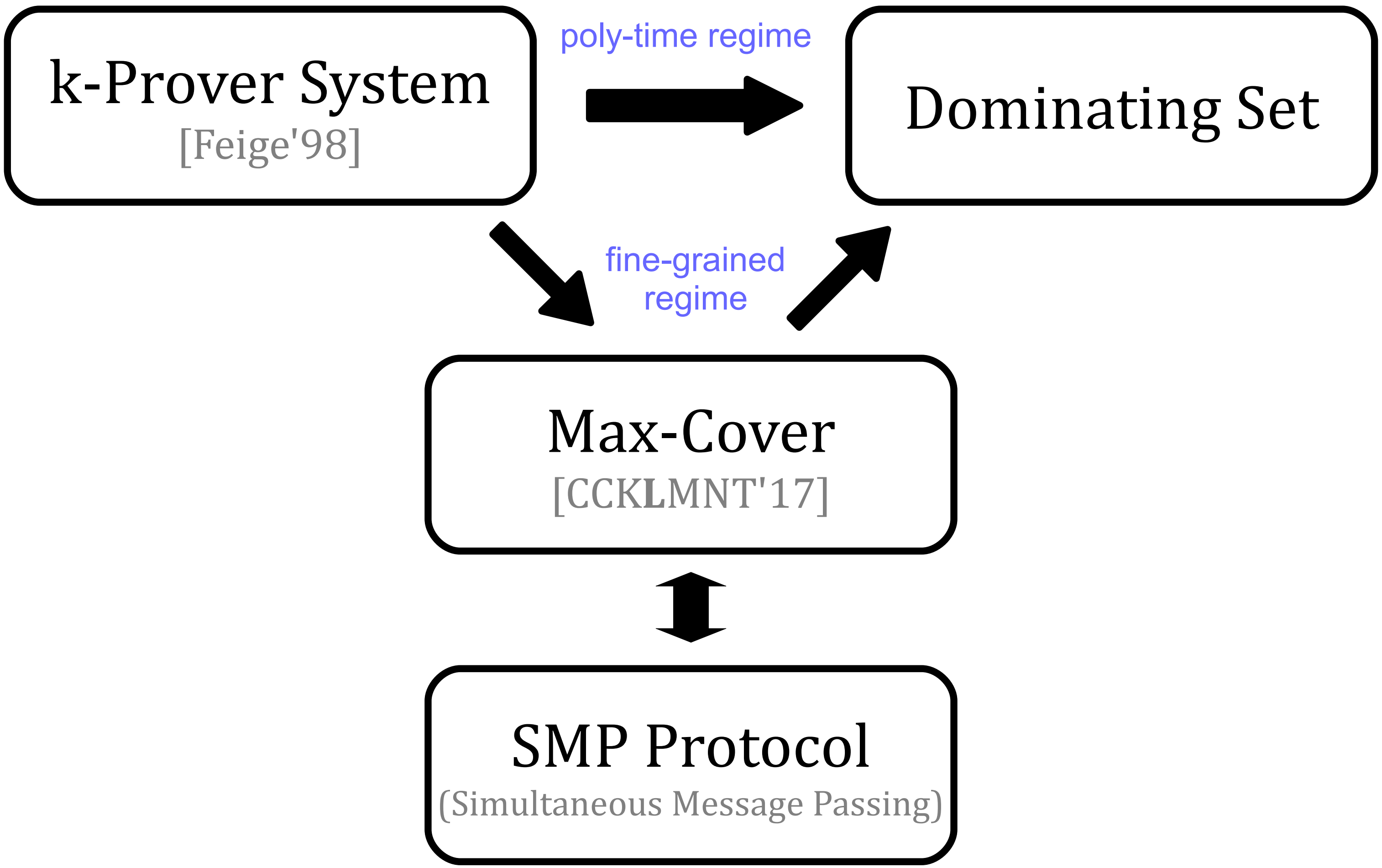

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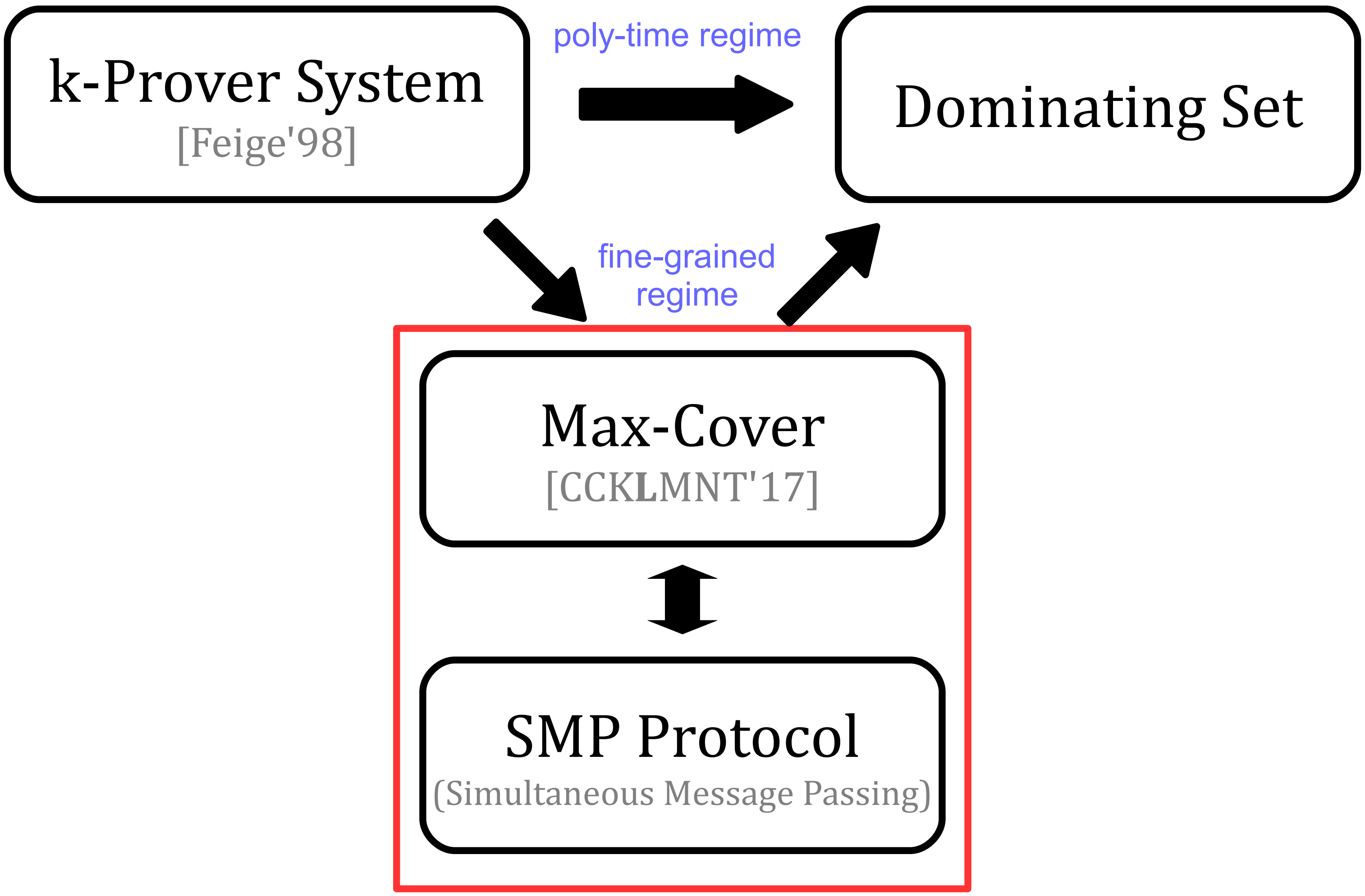
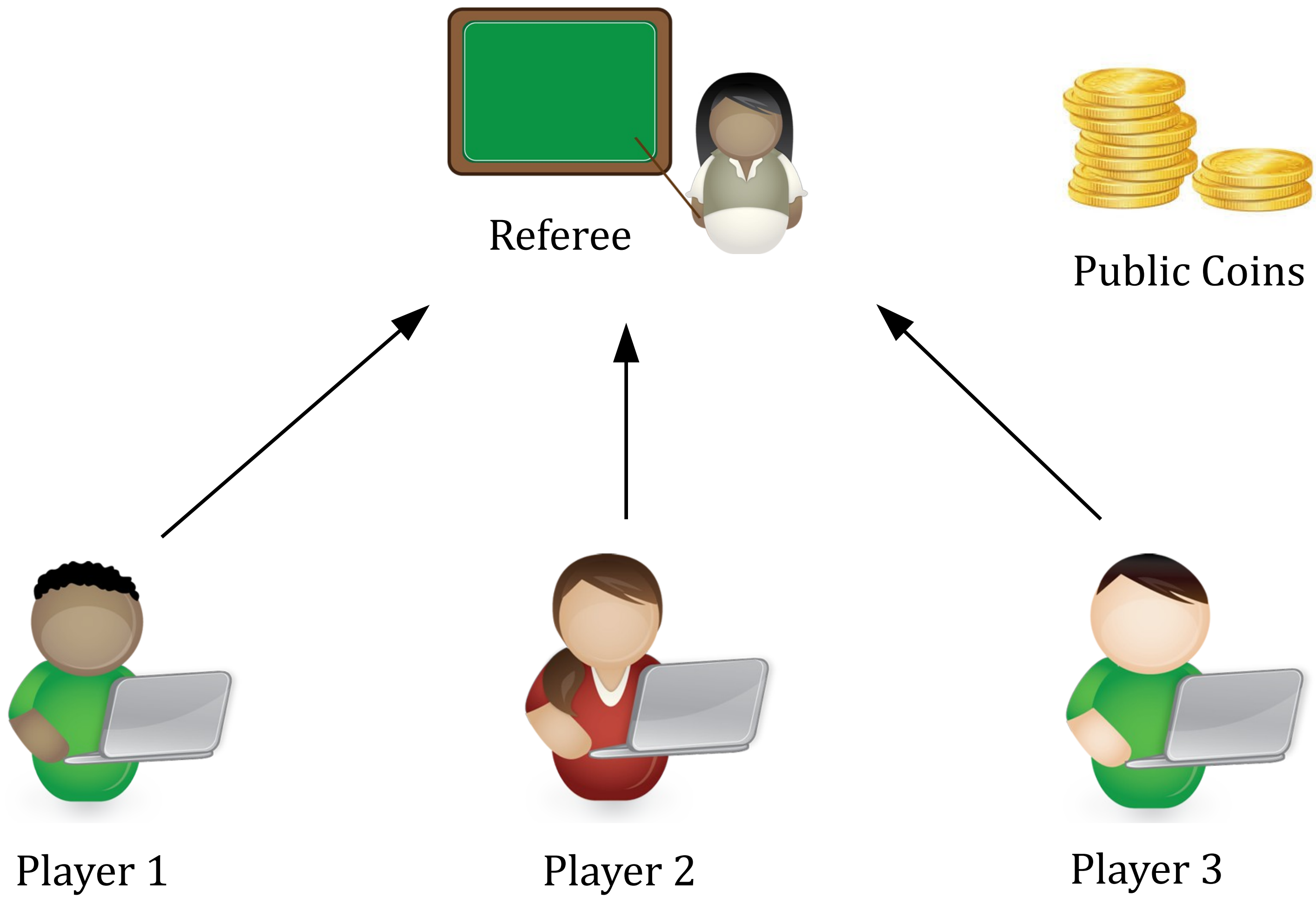


Single Message Passing Protocol



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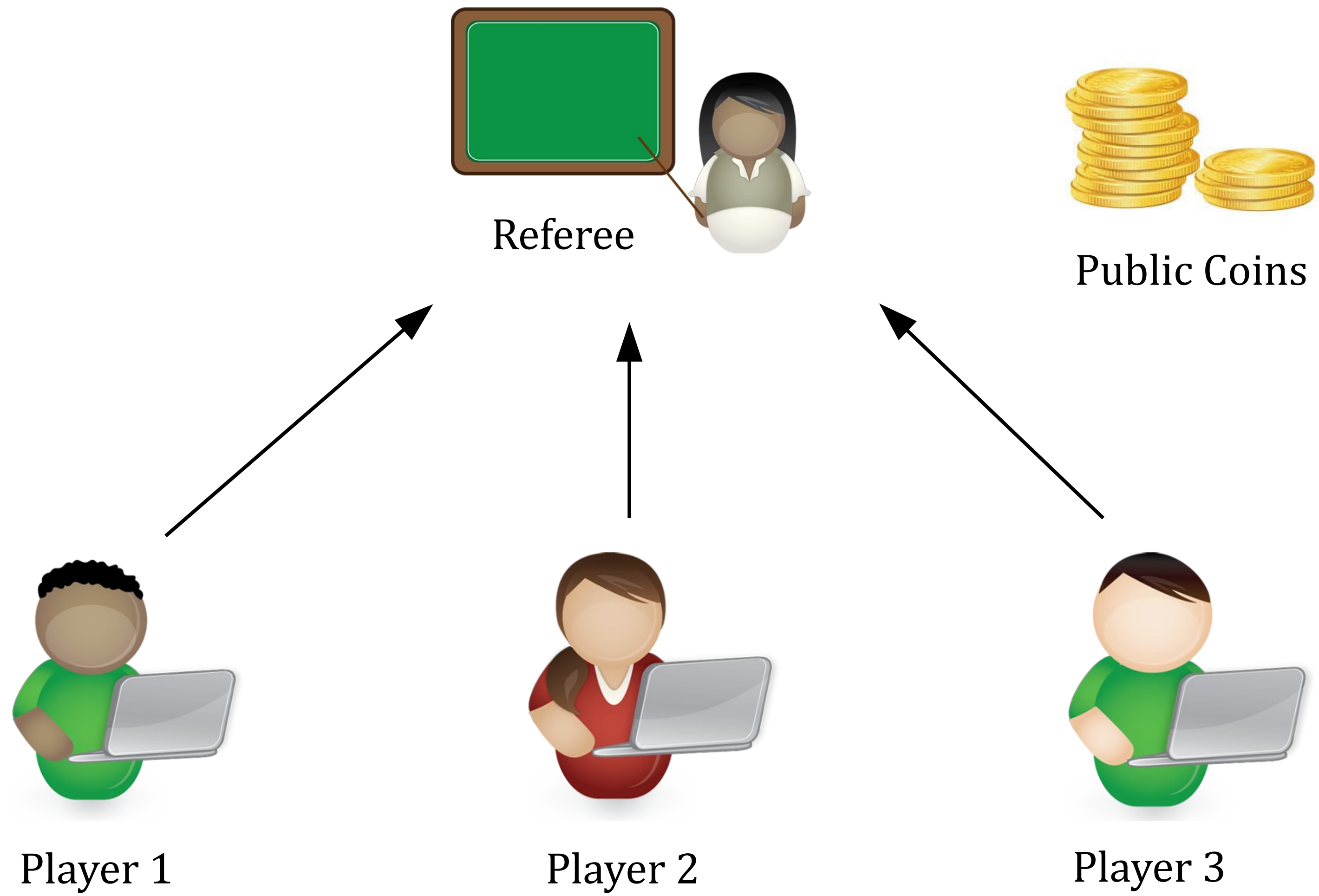
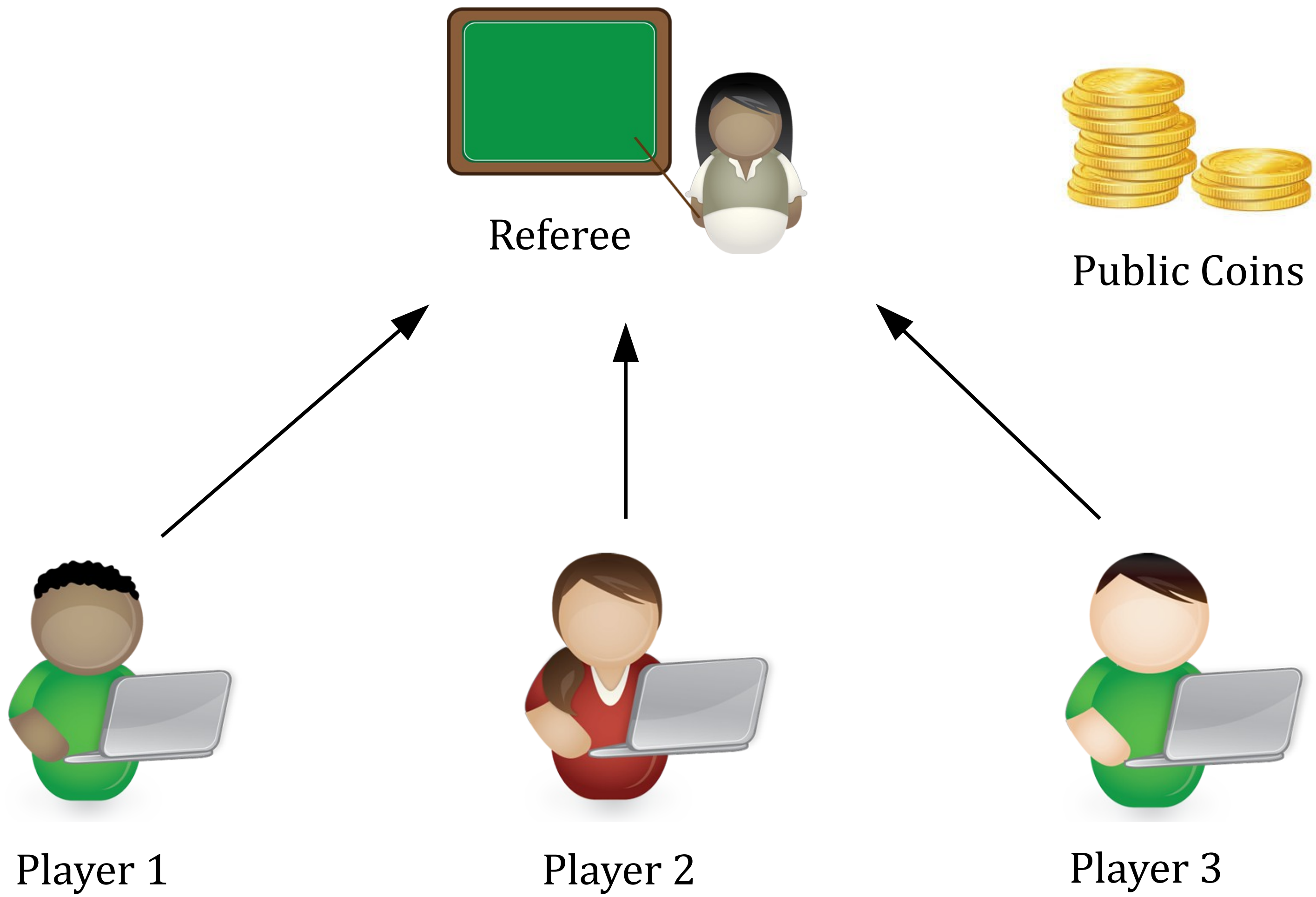


The goal is to use the smallest communication bits.

Single Message Passing Protocol



For our purpose, we want to use $\leq O(1)$ bits.

Known SMP Protocols

that captures popular conjectures

Hypothesis	Protocols	Running Time	
W[1]≠FPT	Agreement	FPT-Time	(T(k) poly(n))
ETH	Set Disjointness	$n^{o(k)}$	
SETH	Set Disjointness	$n^{k-\varepsilon}$	for any $\varepsilon > 0$
k-SUM	Sum-Zero	$n^{k/2-\varepsilon}$	for any $\varepsilon > 0$

All protocols give inapprox ratio $\Omega(\log n)^{1/\mathrm{poly}(k)}$.

Popular Hypotheses

- **ETH:** SAT has no $2^{o(n)}$-time algorithm.
- **SETH:** SAT has no $2^{(1-\varepsilon)n}$-time algorithm.
- **k-SUM:** Find k integers from $S \subseteq [-n^2, n^2]$ that sum to zero requires $n^{k/2-\varepsilon}$-time.
- **FPT ≠ W[1]:** k-DS has no T(k) poly(n)-time algorithm.

FPT≠W[1] Hypothesis

NP-hardness-like assumption in Parameterized Complexity

"k-Clique has no T(k) poly(n) time algorithm."

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"k-Clique has no T(k) poly(n) time algorithm."

FPT is a class of problems that have T(k) poly(n) time (i.e., FPT) algorithms.
W[1]-complete is a class of problems that have no known FPT algorithm.
It is known that FPT ⊆ W[1], but the converse is believed to be false.

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Like SAT is a representative of NP-complete problems ,
k-Clique is a representative of W[1]-complete.

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SMP-Protocol $\leftrightarrow$ Max-Cover $\rightarrow$ k-DS

We will skip the formal definition of
the **S**imultaneous **M**essages **P**assing protocol.

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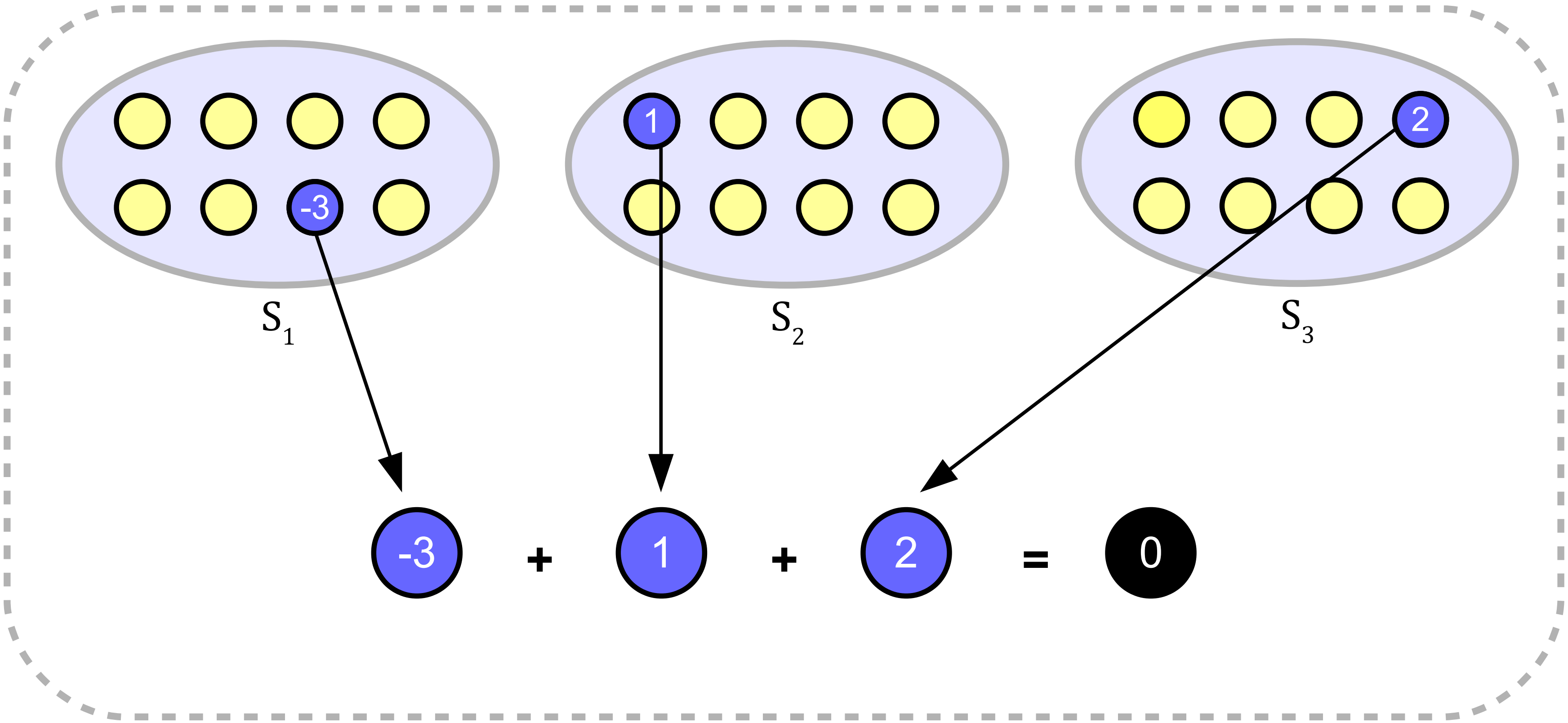
k-SUM (multi-set version)

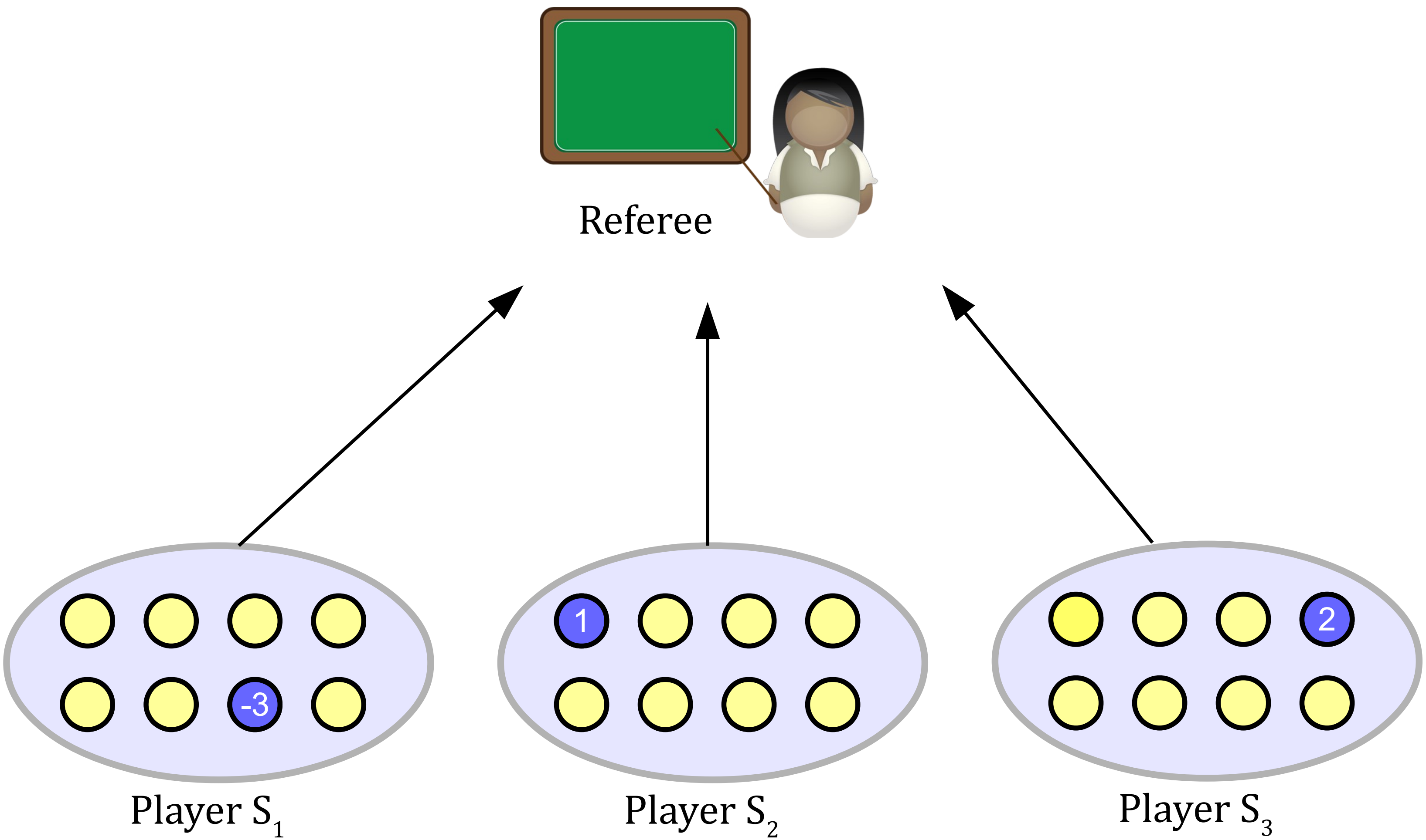
Input: k collections of Integers $S_1, ..., S_k$.

Goal: find k integers $a_1 \in S_1, ..., a_k \in S_k$ s.t. $a_1 + ... + a_k = 0$.



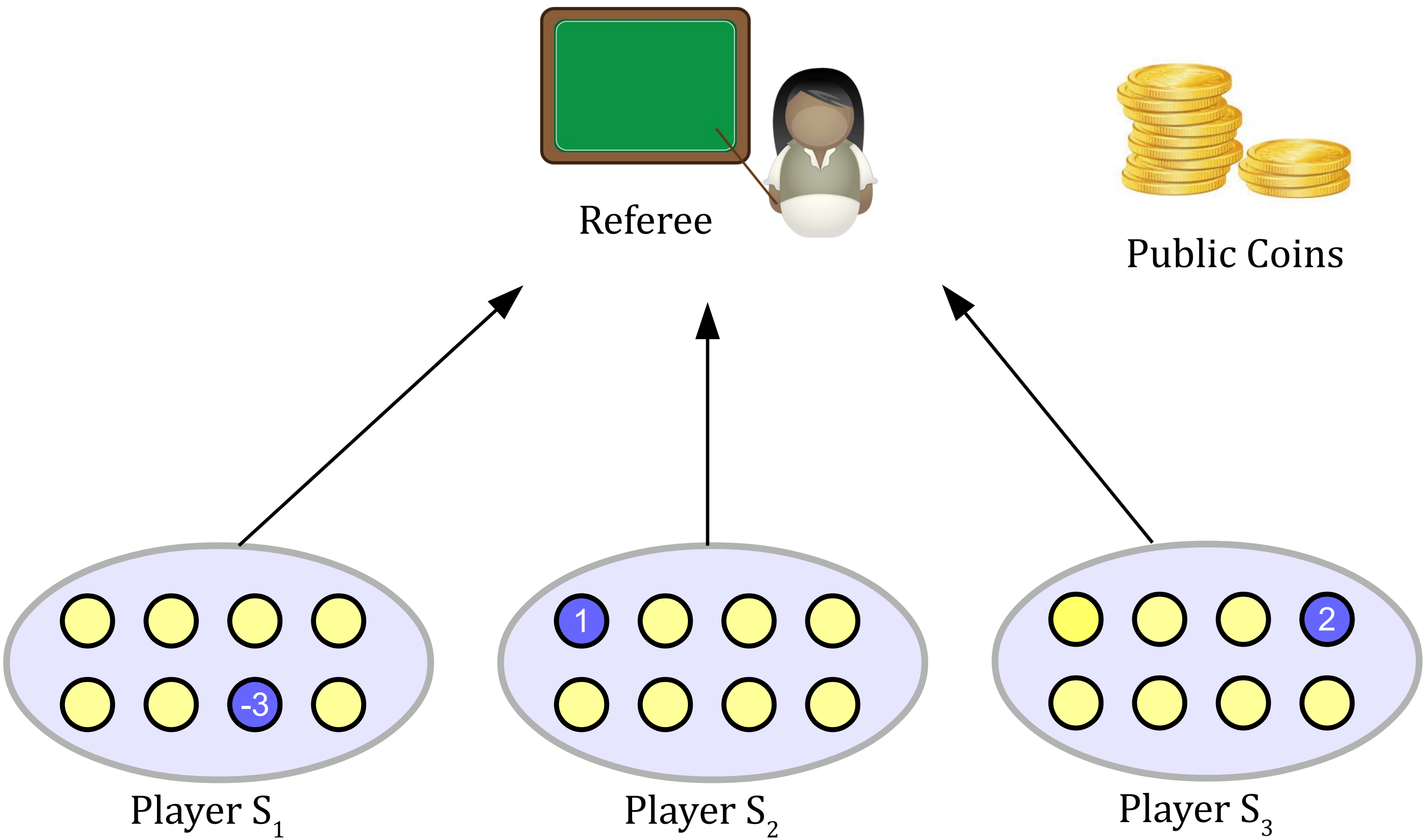
Now we devise a protocol for k-SUM. Let 's assume that each play has one number (presumably given by k-SUM algorithm).

Protocol for k-SUM (Sum-Zero Protocol)



Each player has one number-in-hand (presumably, given by a k-SUM algorithm).
And they, honestly, want to decide if these numbers sum to zero.

Protocol for k-SUM (Sum-Zero Protocol)

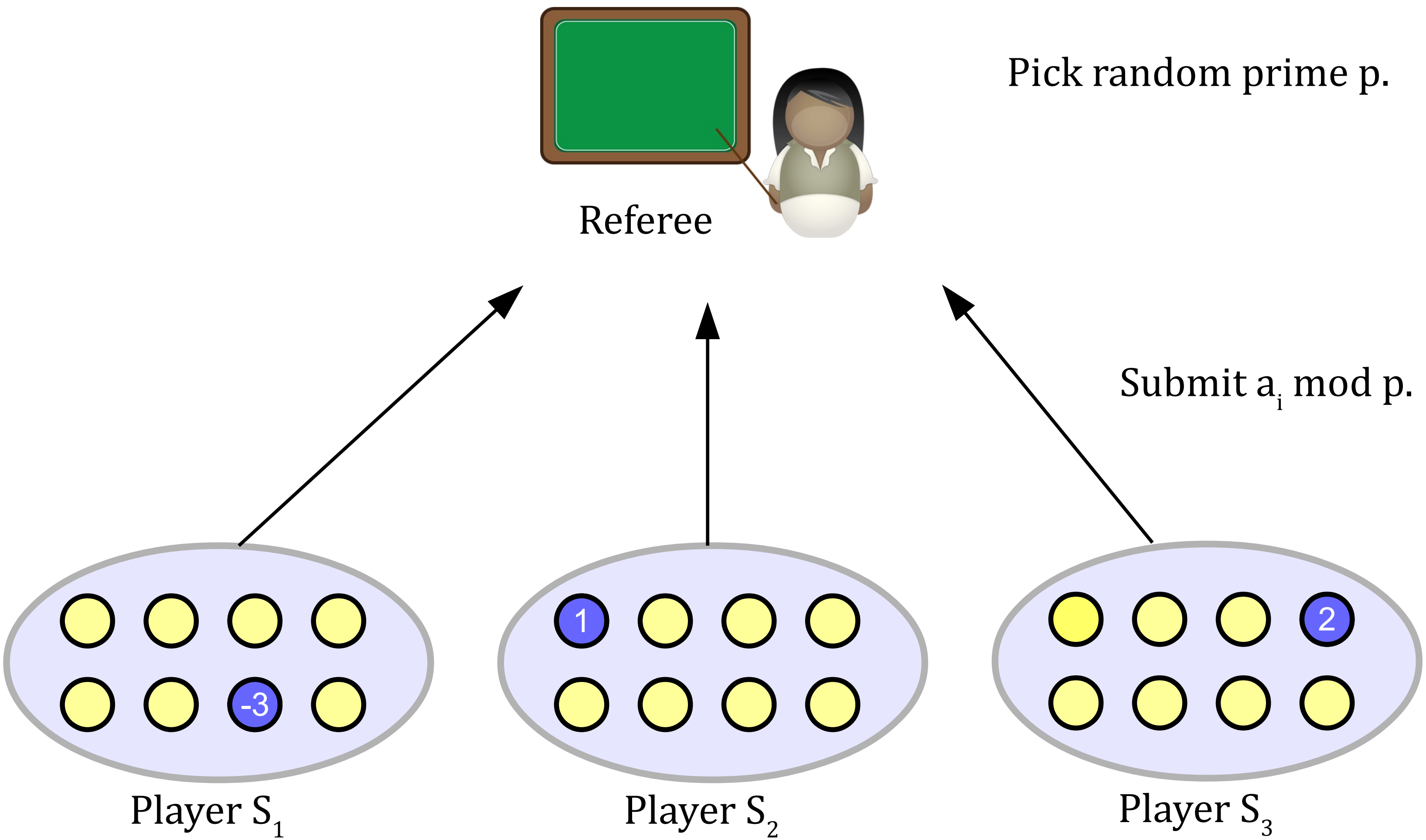


The referee may use randomness to reduce the communication bits.

Protocol for k-SUM (Sum-Zero Protocol)

Pick random prime p.

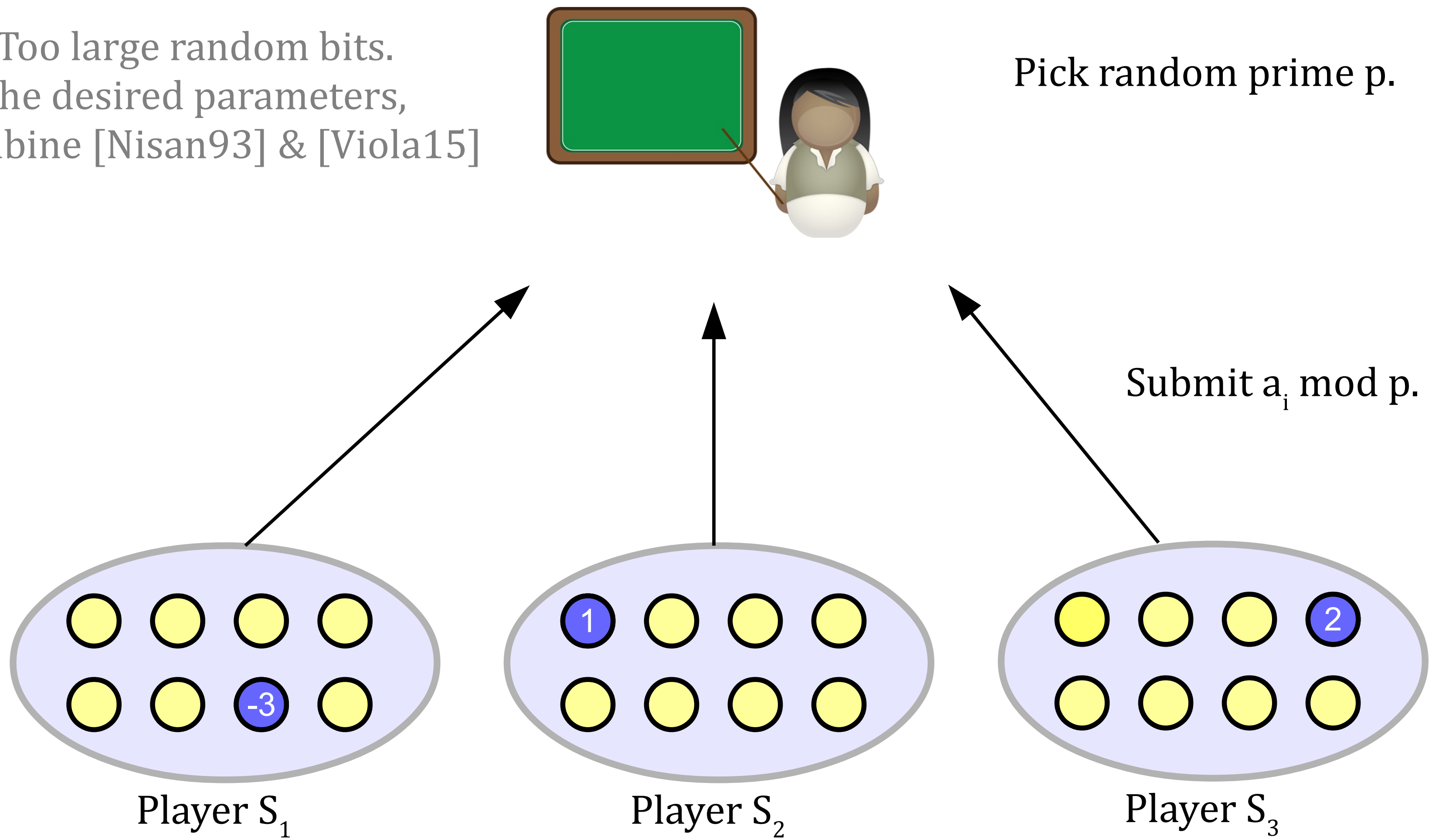
Submit a_i mod p.



$$a_1 + a_2 + a_3 = 0 \;\rightarrow\; (a_1 \bmod p) + (a_2 \bmod p) + (a_3 \bmod p) = 0$$

Protocol for k-SUM (Sum-Zero Protocol)

NOTE: Too large random bits.
To get the desired parameters,
we combine [Nisan93] & [Viola15]



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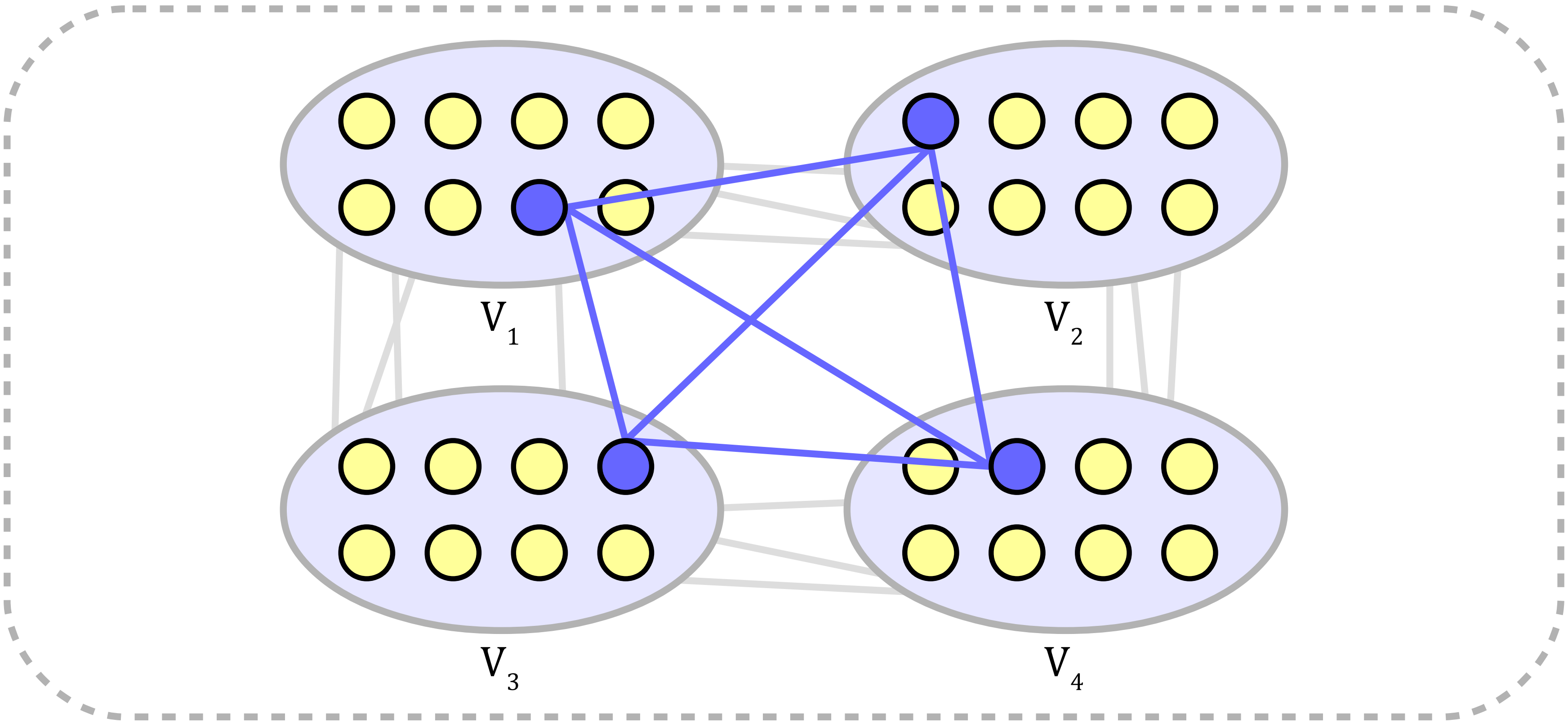
k-Clique (multi-color version)

Input: a bipartite graph $G = (V, E)$, a k-partition of $V = V_1 \uplus \ldots \uplus V_k$

Goal: find a subset of vertices $S \subseteq V(G)$, one from each V_i s.t. $G[S]$ is a clique.

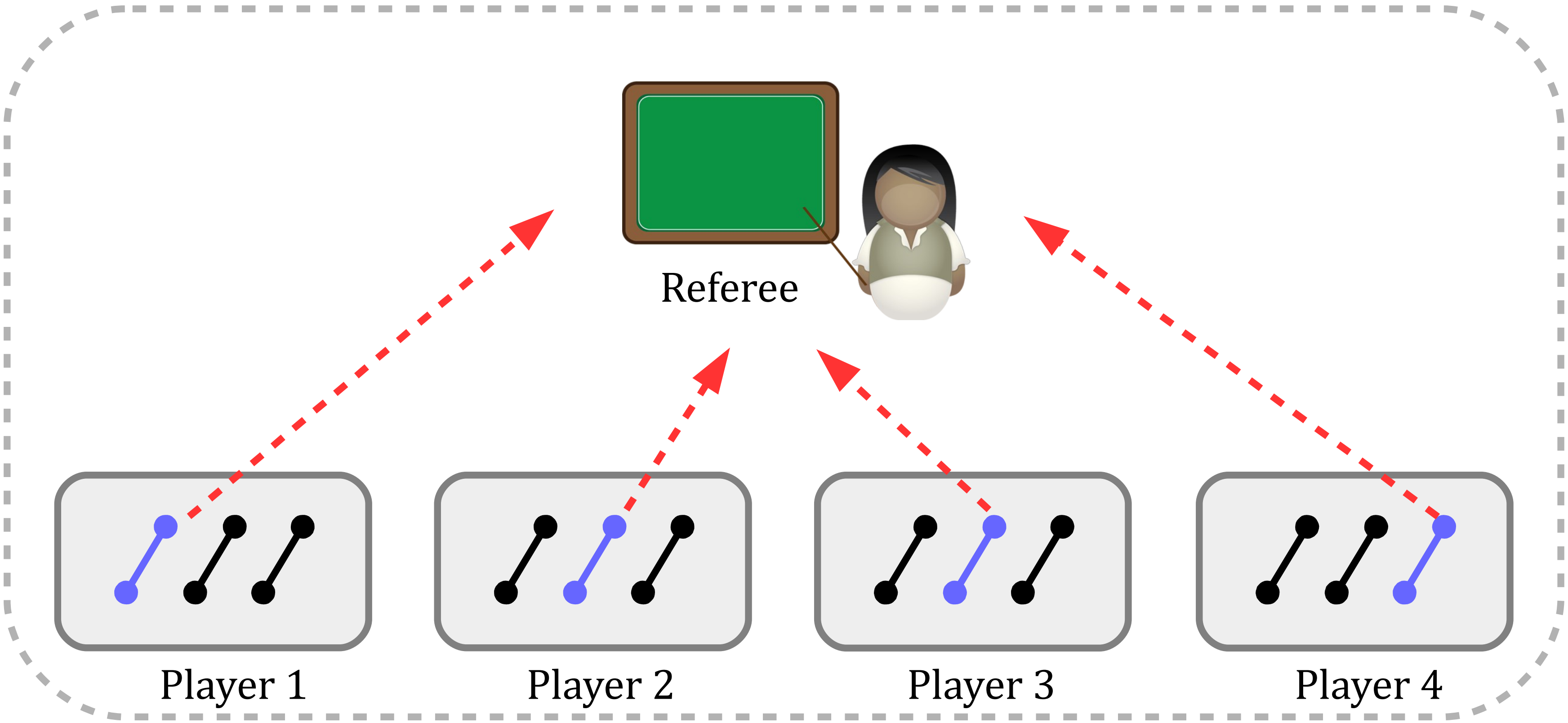


Protocol for k-Clique

(Edge-Vertex Incident Game)

Players: $O(k^2)$ $\{i,j\}$-players, each submits one edge between V_i to V_j.

Referee: Check if there exists k vertices (one from each V_i) such that each vertex is incident to k-1 submitted edges.



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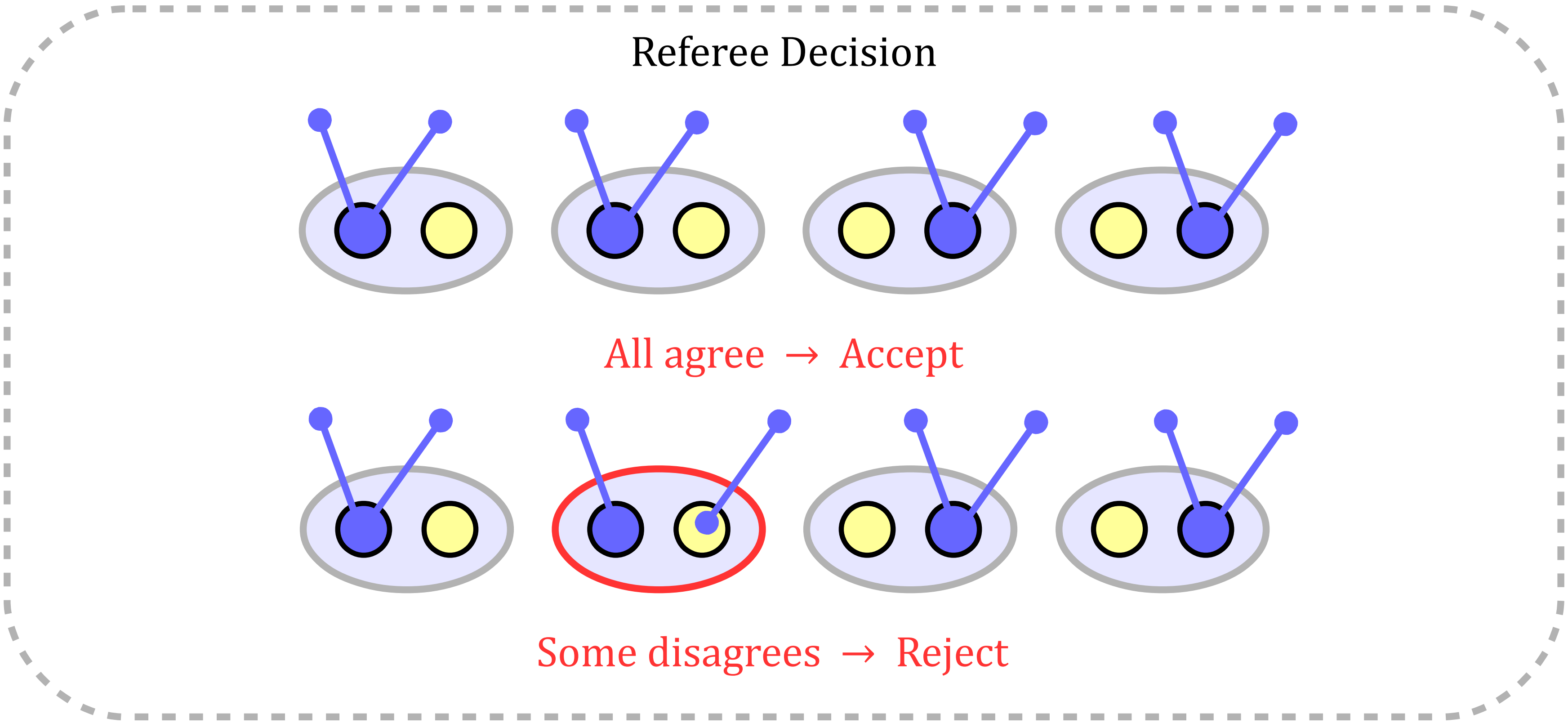

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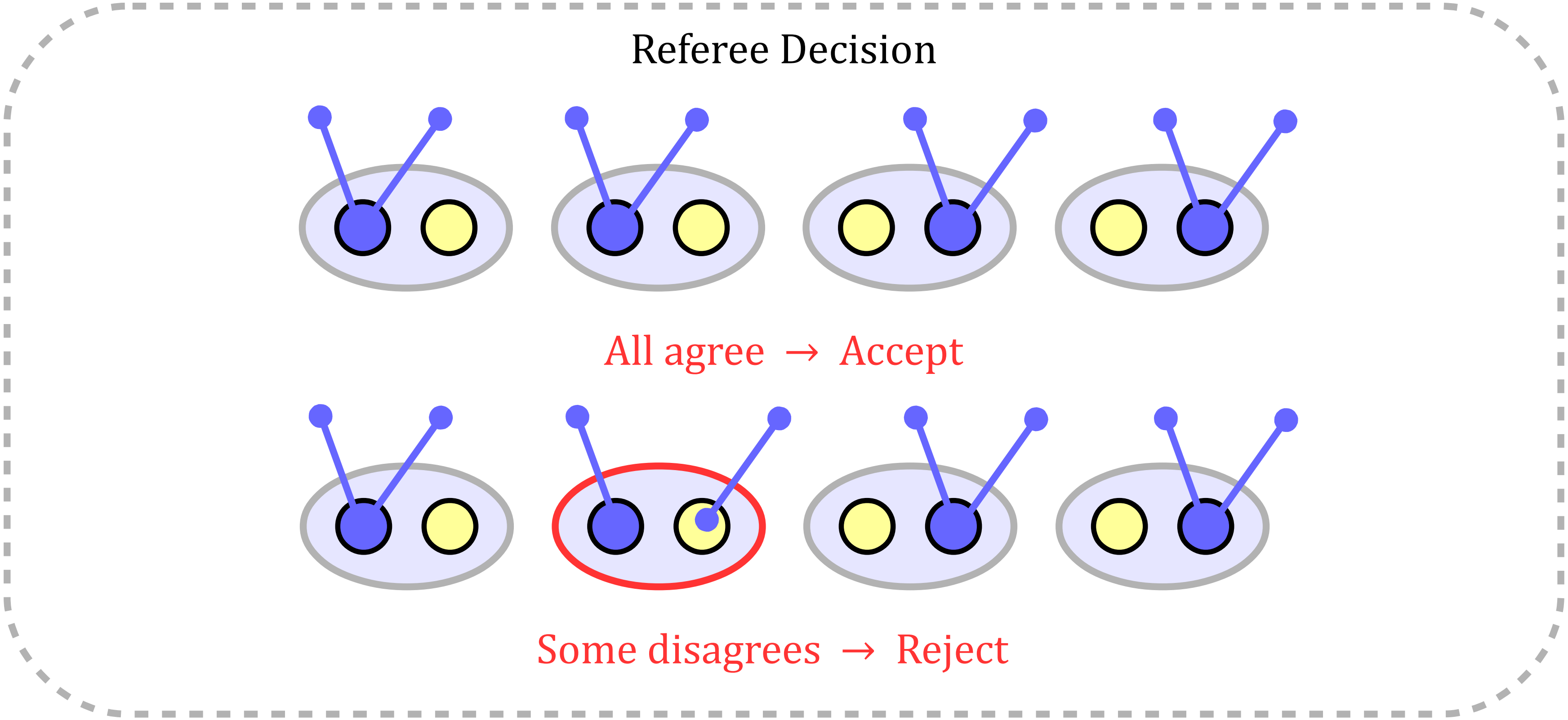


Use **Error Correcting Codes** to reduce communications.

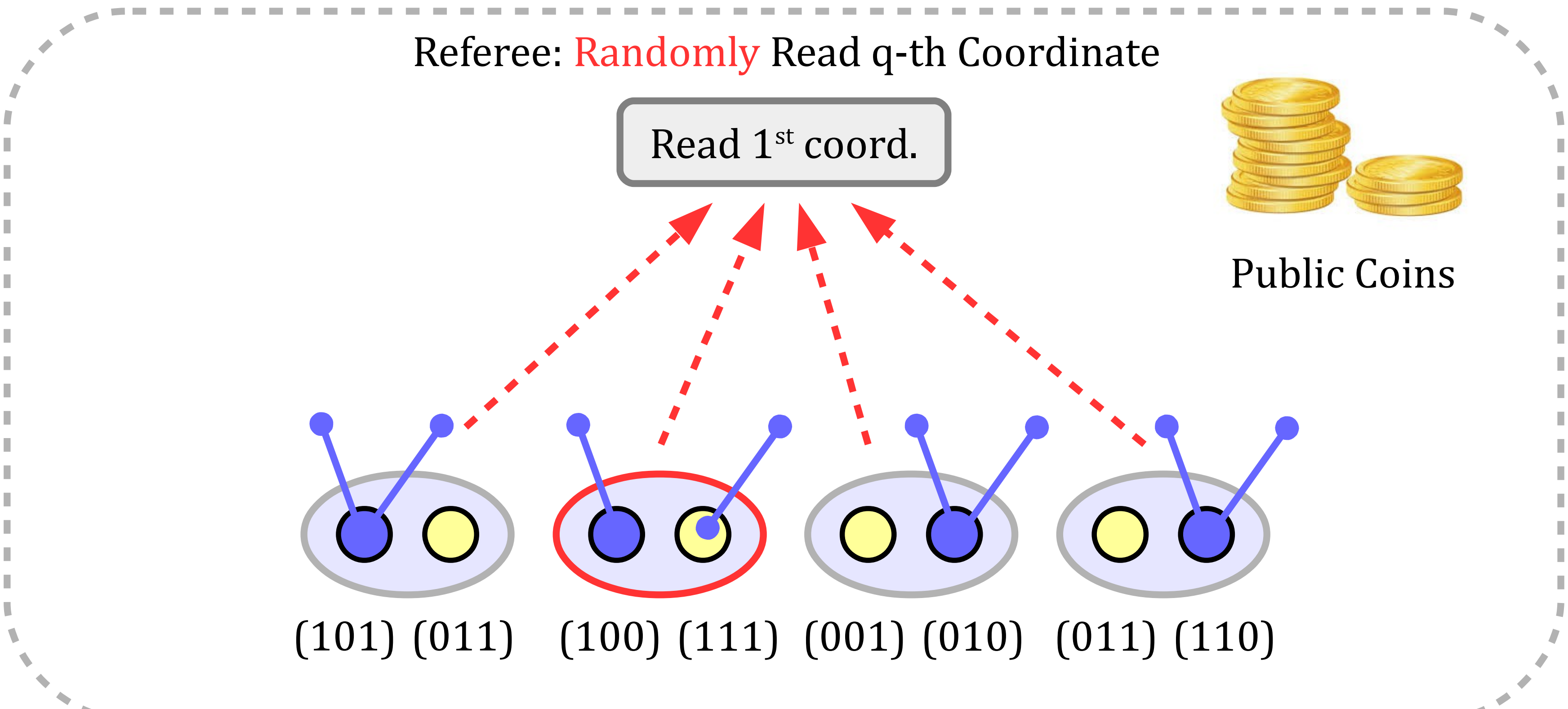
Revised Protocol for k-Clique

(Encode Vertices Using Good Codes)

Codes: Each vertex is encoded with random codes from $[2]^{\log n}$

Players: (i, j)is asked to submit encoded-endpoints of edge V_i - V_j.

Referee: Random an index q. Then check if q-th char of k-1 encoded-edges are consistent.



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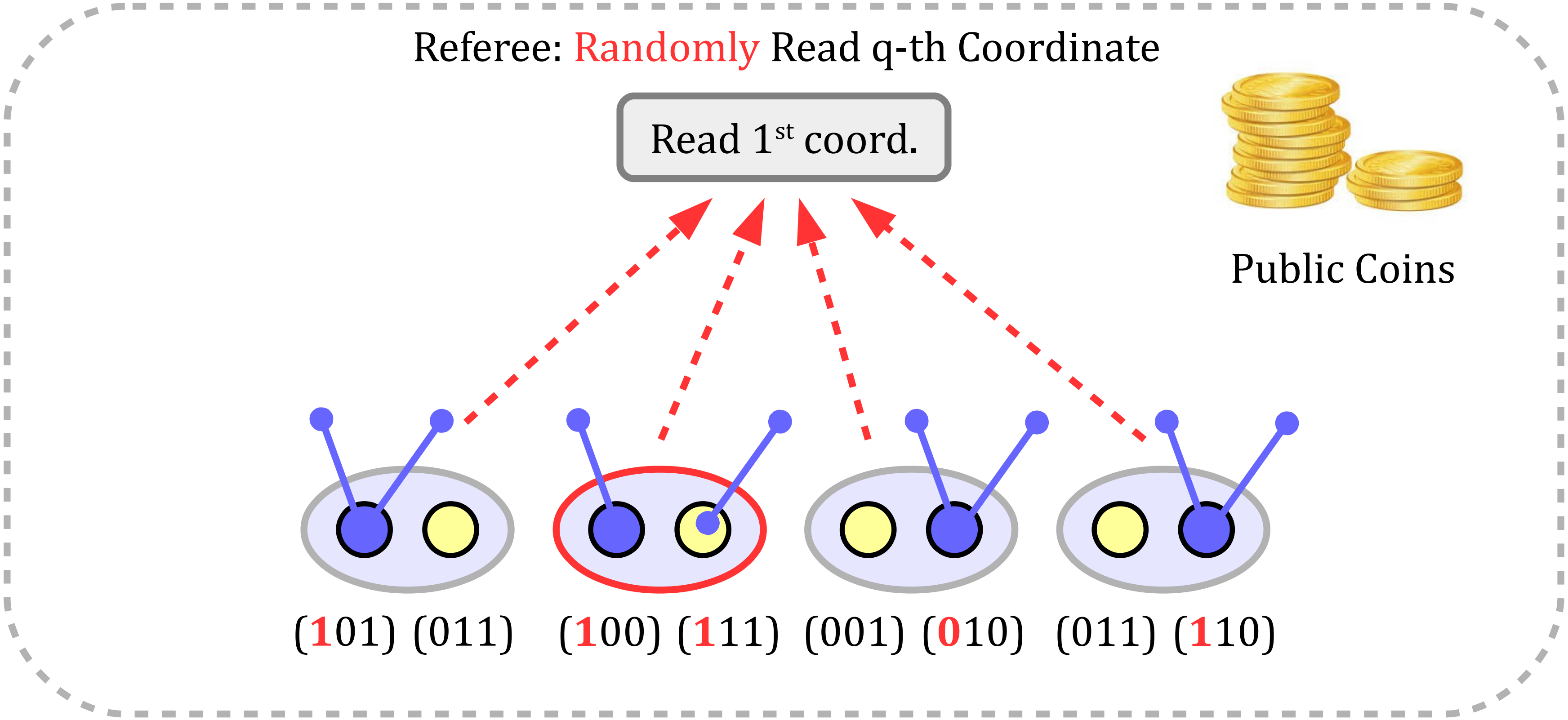

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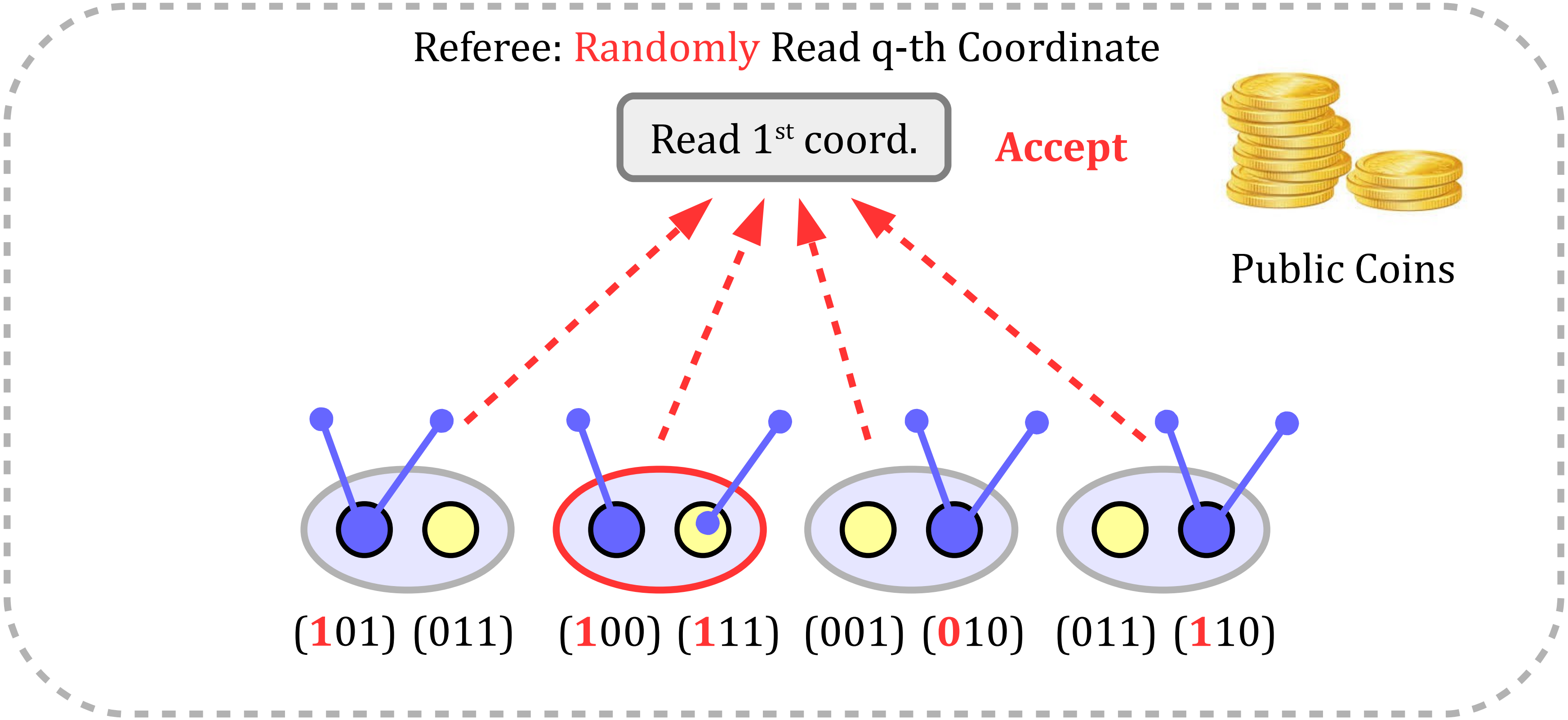

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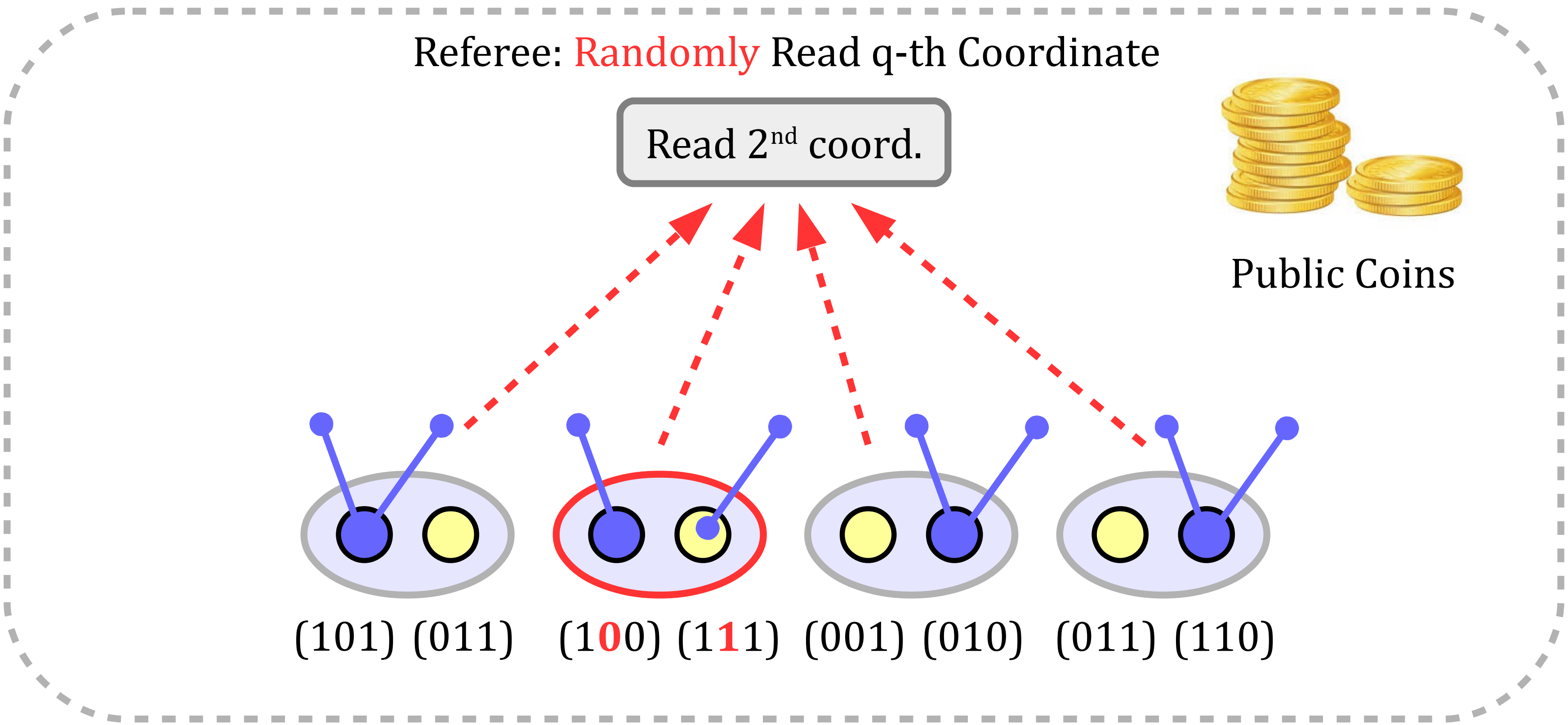

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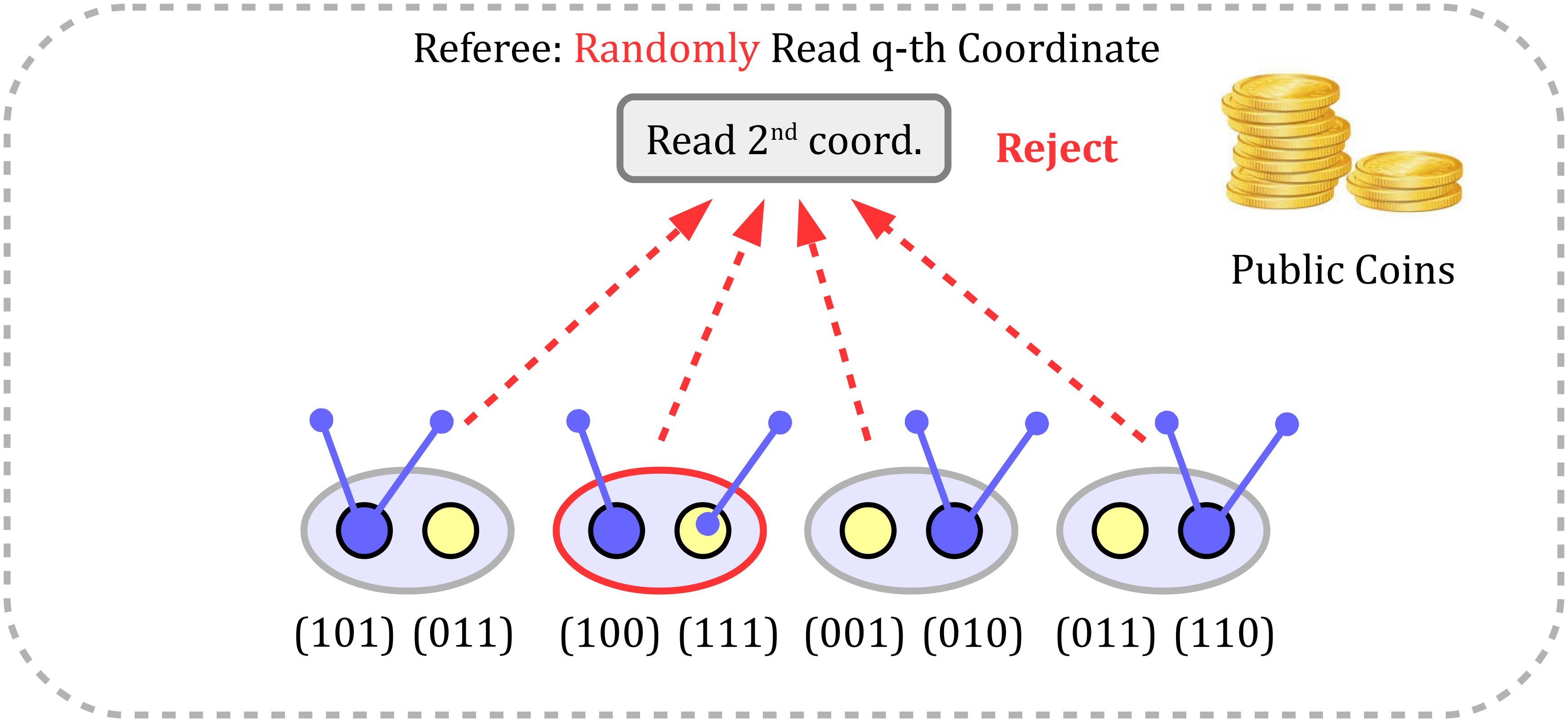

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Conclusion

- Parameterized Inapproximability of k-DS under FPT≠W[1], ETH, SETH, k-SUM.
- Connection between Communication Complexity and Parameterized Inapproximability.
  [SMP-Protocol → Max-Cover → k-DS]

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Recently, Bingkai Lin gives an alternate proof of our results.
His technique is simpler but is not captured by our framework.

Open Problems

- Prove k-DS inapprox under FPT $\neq$ W[2]
  - k-DS is known to be W[2]-complete.
  - Is it possible that it is W[2]-hard even to approximate k-DS?
- Bypassing Gap-ETH for k-Clique
  - FPT-inapprox for k-Clique has been shown under Gap-ETH.
  - Is it possible to base the FPT-inapprox for k-Clique under ETH, SETH or W[1] $\neq$ FPT?

The End

(the real end of this talk)

Thank you for your attention.

Questions?